

Performance Guarantees for ReProCS – Correlated Low-Rank Matrix Entries Case

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Abstract

Online or recursive robust PCA can be posed as a problem of recovering a sparse vector, S_t , and a dense vector, L_t , which lies in a slowly changing low-dimensional subspace, from $M_t := S_t + L_t$ on-the-fly as new data comes in. For initialization, it is assumed that an accurate knowledge of the subspace in which L_0 lies is available. In recent works, Qiu et al proposed and analyzed a novel solution to this problem called recursive projected compressed sensing or ReProCS. In this work, we relax one limiting assumption of Qiu et al's result. Their work required that the L_t 's be mutually independent over time. However this is not a practical assumption, e.g., in the video application, L_t is the background image sequence and one would expect it to be correlated over time. In this work we relax this and allow the L_t 's to follow an autoregressive model. We are able to show that under mild assumptions and under a denseness assumption on the unestimated part of the changed subspace, with high probability (w.h.p.), ReProCS can exactly recover the support set of S_t at all times; the reconstruction errors of both S_t and L_t are upper bounded by a time invariant and small value; and the subspace recovery error decays to a small value within a finite delay of a subspace change time. Because the last assumption depends on an algorithm estimate, this result cannot be interpreted as a correctness result but only a useful step towards it.

I. INTRODUCTION

Principal Components Analysis (PCA) is a widely used dimension reduction technique that finds a small number of orthogonal basis vectors, called principal components (PCs), along which most of the variability of the dataset lies. Often, for time series data, the PCs space changes gradually over time. Updating it on-the-fly (recursively) in the presence of outliers, as more data comes in is referred to as online or recursive robust PCA [1], [2]. As noted in earlier work, an outlier is well modeled as a sparse vector. With this, as will be evident, this problem can also be interpreted as one of recursive sparse recovery in large but structured (low-dimensional) noise.

A key application where the robust PCA problem occurs is in video analysis where the goal is to separate a slowly changing background from moving foreground objects [3], [4]. If we stack each frame as a column vector, the background is well modeled as being dense and lying in a low dimensional subspace that may gradually change over time, while the moving foreground objects constitute the sparse outliers [4]. Other applications include detection of brain activation patterns from functional MRI sequences or detection of anomalous behavior in dynamic networks [5]. There has been a large amount of earlier work on robust PCA, e.g. see [3]. In recent works [4], [6], the batch robust PCA problem has been posed as one of separating a low rank matrix, $\mathcal{L}_t := [L_0, L_1, \dots, L_t]$, from a sparse matrix, $\mathcal{S}_t := [S_0, S_1, \dots, S_t]$, using the measurement matrix, $\mathcal{M}_t := [M_0, M_1, \dots, M_t] = \mathcal{L}_t + \mathcal{S}_t$. A novel convex optimization solution called principal components' pursuit (PCP) has been proposed and shown to achieve exact recovery under mild assumptions. Since then, the batch robust PCA problem, or what is now also often called the sparse+low-rank recovery problem, has been studied extensively. We do not discuss all the works here due to limited space.

Online or recursive robust PCA can be posed as a problem of recovering a sparse vector, S_t , and a dense vector, L_t , which lies in a slowly changing low-dimensional subspace, from $M_t := S_t + L_t$ on-the-fly as new data comes in. For initialization, it is assumed that an accurate knowledge of the subspace in which L_0 lies is available. In recent works, Qiu et al proposed and analyzed a novel solution to this problem called ReProCS [7], [8], [9].

Contribution: In this work, we relax one limiting assumption of Qiu et al's result. Their work required that the L_t 's be mutually independent over time. However this is not a practical assumption, e.g., in the video application, L_t is the background image sequence and one would expect it to be correlated over time. In this work we relax this and allow the L_t 's to follow an autoregressive (AR) model. We are able to show that, under mild assumptions and a denseness assumption on the currently unestimated subspace, with high probability (w.h.p.), ReProCS (with subspace change model known) can exactly recover the support set of S_t at all times; the reconstruction errors of both S_t and L_t are upper bounded by a time invariant and small value; and the subspace recovery error decays to a small value within a finite delay of a subspace change time. The last assumption depends on an algorithm estimate and hence this result also cannot be interpreted as a correctness result but only a useful step towards it.

To the best of our knowledge, the result of Qiu et al and this follow-up work are among the first results for any recursive (online) robust PCA approach, and also for recursive sparse recovery in large but structured (low-dimensional) noise.

Other very recent work on algorithms for recursive / online robust PCA includes [10], [11], [12], [13], [14], [5]. In [13], [14], two online algorithms for robust PCA (that do not model the outlier as a sparse vector but only as a vector that is "far" from the data subspace) have been shown to approximate the batch solution and do so only asymptotically.

II. NOTATION AND BACKGROUND

A. Notation

For a set $T \subset \{1, 2, \dots, n\}$, we use $|T|$ to denote its cardinality, i.e., the number of elements in T . We use T^c to denote its complement w.r.t. $\{1, 2, \dots, n\}$, i.e. $T^c := \{i \in \{1, 2, \dots, n\} : i \notin T\}$.

We use the interval notation, $[t_1, t_2]$, to denote the set of all integers between and including t_1 to t_2 , i.e. $[t_1, t_2] := \{t_1, t_1 + 1, \dots, t_2\}$. For a vector v , v_i denotes the i th entry of v and v_T denotes a vector consisting of the entries of v indexed by T . We use $\|v\|_p$ to denote the ℓ_p norm of v . The support of v , $\text{supp}(v)$, is the set of indices at which v is nonzero, $\text{supp}(v) := \{i : v_i \neq 0\}$. We say that v is s -sparse if $|\text{supp}(v)| \leq s$.

For a matrix B , B' denotes its transpose, and B^\dagger its pseudo-inverse. For a matrix with linearly independent columns, $B^\dagger = (B'B)^{-1}B'$. We use $\|B\|_2 := \max_{x \neq 0} \|Bx\|_2 / \|x\|_2$ to denote the induced 2-norm of the matrix. Also, $\|B\|_*$ is the nuclear norm (sum of singular values) and $\|B\|_{\max}$ denotes the maximum over the absolute values of all its entries. We let $\sigma_i(B)$ denotes the i th largest singular value of B . For a Hermitian matrix, B , we use the notation $B \stackrel{EVD}{=} U\Lambda U'$ to denote the eigenvalue decomposition of B . Here U is an orthonormal matrix and Λ is a diagonal matrix with entries arranged in decreasing order. Also, we use $\lambda_i(B)$ to denote the i th largest eigenvalue of a Hermitian matrix B and we use $\lambda_{\max}(B)$ and $\lambda_{\min}(B)$ denote its maximum and minimum eigenvalues. If B is Hermitian positive semi-definite (p.s.d.), then $\lambda_i(B) = \sigma_i(B)$. For Hermitian matrices B_1 and B_2 , the notation $B_1 \preceq B_2$ means that $B_2 - B_1$ is p.s.d. Similarly, $B_1 \succeq B_2$ means that $B_1 - B_2$ is p.s.d.

For a Hermitian matrix B , $\|B\|_2 = \sqrt{\max(\lambda_{\max}^2(B), \lambda_{\min}^2(B))}$ and thus, $\|B\|_2 \leq b$ implies that $-b \leq \lambda_{\min}(B) \leq \lambda_{\max}(B) \leq b$.

We use I to denote an identity matrix of appropriate size. For an index set T and a matrix B , B_T is the sub-matrix of B containing columns with indices in the set T . Notice that $B_T = BI_T$. Given a matrix B of size $m \times n$ and B_2 of size $m \times n_2$, $[B \ B_2]$ constructs a new matrix by concatenating matrices B and B_2 in the horizontal direction. Let B_{rem} be a matrix containing some columns of B . Then $B \setminus B_{\text{rem}}$ is the matrix B with columns in B_{rem} removed.

For a tall matrix P , $\text{span}(P)$ denotes the subspace spanned by the column vectors of P .

The notation $[\]$ denotes an empty matrix.

Definition 2.1: We refer to a tall matrix P as a *basis matrix* if it satisfies $P'P = I$.

Definition 2.2: We use the notation $Q = \text{basis}(B)$ to mean that Q is a basis matrix and $\text{span}(Q) = \text{span}(B)$. In other words, the columns of Q form an orthonormal basis for the range of B .

Definition 2.3: The s -restricted isometry constant (RIC) [15], δ_s , for an $n \times m$ matrix Ψ is the smallest real number satisfying $(1 - \delta_s)\|x\|_2^2 \leq \|\Psi_T x\|_2^2 \leq (1 + \delta_s)\|x\|_2^2$ for all sets $T \subseteq \{1, 2, \dots, n\}$ with $|T| \leq s$ and all real vectors x of length $|T|$.

It is easy to see that $\max_{T:|T|\leq s} \|(\Psi_T' \Psi_T)^{-1}\|_2 \leq \frac{1}{1-\delta_s(\Psi)}$ [15].

Definition 2.4: We give some notation for random variables in this definition.

- 1) We let $\mathbf{E}[Z]$ denote the expectation of a random variable (r.v.) Z and $\mathbf{E}[Z|X]$ denote its conditional expectation given another r.v. X .
- 2) Let \mathcal{B} be a set of values that a r.v. Z can take. We use \mathcal{B}^e to denote the *event* $Z \in \mathcal{B}$, i.e. $\mathcal{B}^e := \{Z \in \mathcal{B}\}$.
- 3) The probability of any event \mathcal{B}^e can be expressed as [16],

$$\mathbf{P}(\mathcal{B}^e) := \mathbf{E}[\mathbb{I}_{\mathcal{B}}(Z)].$$

where

$$\mathbb{I}_{\mathcal{B}}(Z) := \begin{cases} 1 & \text{if } Z \in \mathcal{B} \\ 0 & \text{otherwise} \end{cases}$$

is the indicator function on the set \mathcal{B} .

- 4) For two events $\mathcal{B}^e, \tilde{\mathcal{B}}^e$, $\mathbf{P}(\mathcal{B}^e|\tilde{\mathcal{B}}^e)$ refers to the conditional probability of \mathcal{B}^e given $\tilde{\mathcal{B}}^e$, i.e. $\mathbf{P}(\mathcal{B}^e|\tilde{\mathcal{B}}^e) := \mathbf{P}(\mathcal{B}^e, \tilde{\mathcal{B}}^e)/\mathbf{P}(\tilde{\mathcal{B}}^e)$.
- 5) For a r.v. X , and a set \mathcal{B} of values that the r.v. Z can take, the notation $\mathbf{P}(\mathcal{B}^e|X)$ is defined as

$$\mathbf{P}(\mathcal{B}^e|X) := \mathbf{E}[\mathbb{I}_{\mathcal{B}}(Z)|X].$$

Notice that $\mathbf{P}(\mathcal{B}^e|X)$ is a r.v. (it is a function of the r.v. X) that always lies between zero and one.

Finally, RHS refers to the right hand side of an equation or inequality; w.p. means “with probability”; and w.h.p. means “with high probability”.

B. Compressive Sensing result

The error bound for noisy compressive sensing (CS) based on the RIC is as follows [17].

Theorem 2.5 ([17]): Suppose we observe

$$y := \Psi x + z$$

where z is the noise. Let \hat{x} be the solution to following problem

$$\min_x \|x\|_1 \text{ subject to } \|y - \Psi x\|_2 \leq \xi \quad (1)$$

Assume that x is s -sparse, $\|z\|_2 \leq \xi$, and $\delta_{2s}(\Psi) < b(\sqrt{2} - 1)$ for some $0 \leq b < 1$. Then the solution of (1) obeys

$$\|\hat{x} - x\|_2 \leq C_1 \xi$$

$$\text{with } C_1 = \frac{4\sqrt{1+\delta_{2s}(\Psi)}}{1-(\sqrt{2}+1)\delta_{2s}(\Psi)} \leq \frac{4\sqrt{1+b(\sqrt{2}-1)}}{1-(\sqrt{2}+1)b(\sqrt{2}-1)}.$$

Remark 2.6: Notice that if b is small enough, C_1 is a small constant but $C_1 > 1$. For example, if $\delta_{2s}(\Psi) \leq 0.15$, then $C_1 \leq 7$. If $C_1 \xi > \|x\|_2$, the normalized reconstruction error bound would be greater than 1, making the result useless. Hence, (1) gives a small reconstruction error bound only for the small noise case, i.e., the case where $\|z\|_2 \leq \xi \ll \|x\|_2$.

C. Results from linear algebra

Davis and Kahan’s $\sin \theta$ theorem [18] studies the rotation of eigenvectors by perturbation.

Theorem 2.7 ($\sin \theta$ theorem [18]): Given two Hermitian matrices \mathcal{A} and \mathcal{H} satisfying

$$\mathcal{A} = \begin{bmatrix} E & E_{\perp} \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & A_{\perp} \end{bmatrix} \begin{bmatrix} E' \\ E'_{\perp} \end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} E & E_{\perp} \end{bmatrix} \begin{bmatrix} H & B' \\ B & H_{\perp} \end{bmatrix} \begin{bmatrix} E' \\ E'_{\perp} \end{bmatrix} \quad (2)$$

where $\begin{bmatrix} E & E_{\perp} \end{bmatrix}$ is an orthonormal matrix. The two ways of representing $\mathcal{A} + \mathcal{H}$ are

$$\mathcal{A} + \mathcal{H} = \begin{bmatrix} E & E_{\perp} \end{bmatrix} \begin{bmatrix} A+H & B' \\ B & A_{\perp} + H_{\perp} \end{bmatrix} \begin{bmatrix} E' \\ E'_{\perp} \end{bmatrix} = \begin{bmatrix} F & F_{\perp} \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda_{\perp} \end{bmatrix} \begin{bmatrix} F' \\ F'_{\perp} \end{bmatrix}$$

where $[F \ F_\perp]$ is another orthonormal matrix. Let $\mathcal{R} := (\mathcal{A} + \mathcal{H})E - \mathcal{A}E = \mathcal{H}E$. If $\lambda_{\min}(A) > \lambda_{\max}(\Lambda_\perp)$, then

$$\|(I - FF')E\|_2 \leq \frac{\|\mathcal{R}\|_2}{\lambda_{\min}(A) - \lambda_{\max}(\Lambda_\perp)}$$

The above result bounds the amount by which the two subspaces $\text{span}(E)$ and $\text{span}(F)$ differ as a function of the norm of the perturbation $\|\mathcal{R}\|_2$ and of the gap between the minimum eigenvalue of A and the maximum eigenvalue of Λ_\perp . Next, we state Weyl's theorem which bounds the eigenvalues of a perturbed Hermitian matrix, followed by Ostrowski's theorem.

Theorem 2.8 (Weyl [19]): Let \mathcal{A} and \mathcal{H} be two $n \times n$ Hermitian matrices. For each $i = 1, 2, \dots, n$ we have

$$\lambda_i(\mathcal{A}) + \lambda_{\min}(\mathcal{H}) \leq \lambda_i(\mathcal{A} + \mathcal{H}) \leq \lambda_i(\mathcal{A}) + \lambda_{\max}(\mathcal{H})$$

Theorem 2.9 (Ostrowski [19]): Let H and W be $n \times n$ matrices, with H Hermitian and W nonsingular. For each $i = 1, 2, \dots, n$, there exists a positive real number θ_i such that $\lambda_{\min}(WW') \leq \theta_i \leq \lambda_{\max}(WW')$ and $\lambda_i(WHW') = \theta_i \lambda_i(H)$. Therefore,

$$\lambda_{\min}(WHW') \geq \lambda_{\min}(WW')\lambda_{\min}(H)$$

The following lemma proves some simple linear algebra facts.

Lemma 2.10: Suppose that P , \hat{P} and Q are three basis matrices. Also, P and \hat{P} are of the same size, $Q'P = 0$ and $\|(I - \hat{P}\hat{P}')P\|_2 = \zeta_*$. Then,

- 1) $\|(I - \hat{P}\hat{P}')PP'\|_2 = \|(I - PP')\hat{P}\hat{P}'\|_2 = \|(I - PP')\hat{P}\|_2 = \|(I - \hat{P}\hat{P}')P\|_2 = \zeta_*$
- 2) $\|PP' - \hat{P}\hat{P}'\|_2 \leq 2\|(I - \hat{P}\hat{P}')P\|_2 = 2\zeta_*$
- 3) $\|\hat{P}'Q\|_2 \leq \zeta_*$
- 4) $\sqrt{1 - \zeta_*^2} \leq \sigma_i((I - \hat{P}\hat{P}')Q) \leq 1$

The proof is in the Appendix.

D. High probability tail bounds for sums of random matrices

The following lemma follows easily using Definition 2.4. We will use this at various places in the paper.

Lemma 2.11: Suppose that \mathcal{B} is the set of values that the r.v.s X, Y can take. Suppose that \mathcal{C} is a set of values that the r.v. X can take. For a $0 \leq p \leq 1$, if $\mathbf{P}(\mathcal{B}^e|X) \geq p$ for all $X \in \mathcal{C}$, then $\mathbf{P}(\mathcal{B}^e|\mathcal{C}^e) \geq p$ as long as $\mathbf{P}(\mathcal{C}^e) > 0$.

The proof is in the Appendix.

The following lemma is an easy consequence of the chain rule of probability applied to a contracting sequence of events.

Lemma 2.12: For a sequence of events $E_0^e, E_1^e, \dots, E_m^e$ that satisfy $E_0^e \supseteq E_1^e \supseteq E_2^e \dots \supseteq E_m^e$, the following holds

$$\mathbf{P}(E_m^e|E_0^e) = \prod_{k=1}^m \mathbf{P}(E_k^e|E_{k-1}^e).$$

Proof: $\mathbf{P}(E_m^e|E_0^e) = \mathbf{P}(E_m^e, E_{m-1}^e, \dots, E_0^e|E_0^e) = \prod_{k=1}^m \mathbf{P}(E_k^e|E_{k-1}^e, E_{k-2}^e, \dots, E_0^e) = \prod_{k=1}^m \mathbf{P}(E_k^e|E_{k-1}^e)$. ■

Next, we state the matrix Azuma inequality which gives tail bounds for sums of random matrices.

Theorem 2.13: (Matrix Azuma) Consider a finite adapted sequence $\{Z_t\}_{t \geq 1}$ of Hermitian matrices in dimension n , and a fixed sequence A_t of self-adjoint matrices that satisfy

$$\mathbf{E}_{t-1}(Z_t) = \mathbf{E}(Z_t|Z_1, \dots, Z_{t-1}) = 0, \text{ and } Z_t^2 \preceq A_t^2 \text{ almost surely.}$$

Then, for all $\epsilon > 0$,

$$\mathbf{P}\left(\lambda_{\max}\left(\sum_t Z_t\right) \leq \epsilon\right) \geq 1 - n \exp\left(-\frac{\epsilon^2}{8\sigma^2}\right), \text{ where } \sigma^2 = \left\|\sum_t A_t^2\right\|.$$

The proof is in the appendix.

Corollary 2.14 (Matrix Azuma conditioned on another random variable for any zero mean matrix): Consider an α -length adapted sequence $\{Z_t\}$ of random matrices of size $n_1 \times n_2$ given a random variable X . Assume that, for all $X \in \mathcal{C}$, (i)

$\mathbf{P}(\|Z_t\|_2 \leq b_1|X) = 1$ and (ii) $\mathbf{E}_{t-1}(Z_t|X) = 0$. Then, for all $\epsilon > 0$,

$$\mathbf{P}\left(\left\|\frac{1}{\alpha}\sum_{t=1}^{\alpha}Z_t\right\|\leq\epsilon|X\right)\geq 1-(n_1+n_2)\exp\left(-\frac{\alpha\epsilon^2}{32b_1^2}\right)$$

The proof is in the appendix.

Corollary 2.15 (Matrix Azuma conditioned on another random variable for a nonzero mean Hermitian matrix): Consider an α -length sequence $\{Z_t\}_{1\leq t\leq\alpha}$ of random Hermitian matrices of size $n\times n$ given a random variable X . Assume that, for all $X\in\mathcal{C}$, (i) $\mathbf{P}(b_1I\preceq Z_t\preceq b_2I|X)=1, 1\leq t\leq\alpha$ and (ii) $b_3I\preceq\frac{1}{\alpha}\sum_{t=1}^{\alpha}\mathbf{E}_{t-1}(Z_t|X)\preceq b_4I$. Then for all $\epsilon>0$,

$$\begin{aligned}\mathbf{P}\left(\lambda_{\max}\left(\frac{1}{\alpha}\sum_{t=1}^{\alpha}Z_t|X\right)\leq b_4+\epsilon\right)&\geq 1-n\exp\left(-\frac{\alpha\epsilon^2}{8(b_2-b_1)^2}\right)\\ \mathbf{P}\left(\lambda_{\min}\left(\frac{1}{\alpha}\sum_{t=1}^{\alpha}Z_t|X\right)\geq b_3-\epsilon\right)&\geq 1-n\exp\left(-\frac{\alpha\epsilon^2}{8(b_2-b_1)^2}\right)\end{aligned}$$

The proof is in the appendix.

Definition 2.16: Let function $F(\alpha, \epsilon, b_2 - b_1) := \exp(-\frac{\alpha\epsilon^2}{8(b_2-b_1)^2})$.

Lemma 2.17: If random variable X and Y are independent, $h(\cdot)$, $g(\cdot)$ are some functions, then $\mathbf{E}(Xh(Y)|g(Y)) = \mathbf{E}(X)\mathbf{E}(h(Y)|g(Y))$.

The proof is in the appendix.

III. PROBLEM DEFINITION

The measurement vector at time t , M_t , satisfies

$$M_t := L_t + S_t \tag{3}$$

where S_t is a sparse vector and L_t is a dense vector that satisfies the model given below. Denote by P_0 a basis matrix for $\mathcal{L}_{t_{\text{train}}} = [L_0, L_1, \dots, L_{t_{\text{train}}}]$, i.e., $\text{span}(P_0) = \text{span}(\mathcal{L}_{t_{\text{train}}})$. We are given an accurate enough estimate \hat{P}_0 for P_0 , i.e., $\|(I - \hat{P}_0\hat{P}_0')P_0\|_2$ is small. The goal is

- 1) to estimate both S_t and L_t at each time $t > t_{\text{train}}$, and
- 2) to estimate $\text{span}(\mathcal{L}_t)$ every so often.

A. Signal Model

Signal Model 3.1 (Model on L_t):

- 1) We assume that $L_t = P_{(t)}a_t$ with $P_{(t)} = P_j$ for all $t_j \leq t < t_{j+1}$, $j = 0, 1, 2, \dots, J$, where P_j is an $n \times r_j$ basis matrix with $r_j \ll \min(n, (t_{j+1} - t_j))$. At the change times, t_j , P_j changes as

$$P_j = [P_{j-1} \ P_{j,\text{new}}].$$

Here, $P_{j,\text{new}}$ is an $n \times c_{j,\text{new}}$ basis matrix with $P_{j,\text{new}}'P_{j-1} = 0$. Thus $r_j = r_{j-1} + c_{j,\text{new}}$. We let $t_0 = 0$ and t_{J+1} equal the sequence length or $t_{J+1} = \infty$.

- 2) The vector of coefficients, $a_t := P_{(t)}'L_t$, satisfies the following autoregressive model

$$a_t = ba_{t-1} + \nu_t$$

where $b < 1$ is a scalar, $\mathbf{E}[\nu_t] = 0$, ν_t 's are mutually independent over time t and the entries of any ν_t are pairwise uncorrelated, i.e. $\mathbf{E}[(\nu_t)_i(\nu_t)_j] = 0$ when $i \neq j$.

Definition 3.2: Define the covariance matrices of ν_t and a_t to be the diagonal matrices $\Lambda_{\nu,t} := \text{Cov}[\nu_t] = \mathbf{E}(\nu_t\nu_t')$ and $\Lambda_{a,t} := \text{Cov}[a_t] = \mathbf{E}(a_t a_t')$. Then clearly,

$$\Lambda_{a,t} = b^2\Lambda_{a,t-1} + \Lambda_{\nu,t} \tag{4}$$

Also, for $t_j \leq t < t_{j+1}$, a_t is an r_j length vector which can be split as

$$a_t = P_j' L_t = \begin{bmatrix} a_{t,*} \\ a_{t,\text{new}} \end{bmatrix}$$

where $a_{t,*} := P_{j-1}' L_t$ is an r_{j-1} length vector and $a_{t,\text{new}} := P_{j,\text{new}}' L_t$ is a $c_{j,\text{new}}$ length vector. Thus, for this interval, L_t can be rewritten as

$$L_t = [P_{j-1} \ P_{j,\text{new}}] \begin{bmatrix} a_{t,*} \\ a_{t,\text{new}} \end{bmatrix} = P_{j-1} a_{t,*} + P_{j,\text{new}} a_{t,\text{new}}$$

Also, $\Lambda_{a,t}$ can be split as

$$\Lambda_{a,t} = \begin{bmatrix} (\Lambda_{a,t})_* & 0 \\ 0 & (\Lambda_{a,t})_{\text{new}} \end{bmatrix}$$

where $(\Lambda_{a,t})_* = \text{Cov}[a_{t,*}]$ and $(\Lambda_{a,t})_{\text{new}} = \text{Cov}[a_{t,\text{new}}]$ are diagonal matrices.

Assumption 3.3: Assume that L_t satisfies Signal Model 3.1 with

- 1) $0 \leq c_{j,\text{new}} \leq c$ for all j (thus $r_j \leq r_{\max} := r_0 + Jc$)
- 2) $\|\nu_t\|_\infty \leq (1-b)\gamma_*$
- 3)

$$\max_j \max_{t \in \mathcal{I}_{j,k}} \|\nu_{t,\text{new}}\|_\infty \leq (1-b)\gamma_{\text{new},k}, \quad (5)$$

where $\gamma_{\text{new},k} = \min(v^{k-1}\gamma_{\text{new}}, \gamma_*)$ with a $v > 1$

4)

$$(1-b^2)\lambda^- \leq \lambda_{\min}(\Lambda_{\nu,t}) \leq \lambda_{\max}(\Lambda_{\nu,t}) \leq (1-b^2)\lambda^+,$$

and

$$\lambda^- \leq \lambda_{\min}(\Lambda_{a,0}) \leq \lambda_{\max}(\Lambda_{a,0}) \leq (1-b^2)\lambda^+,$$

where $0 < \lambda^- < \lambda^+ < \infty$

- 5) $(1-b^2)\lambda_{\text{new}}^- \leq \lambda_{\min}((\Lambda_{\nu,t})_{\text{new}}) \leq \lambda_{\max}((\Lambda_{\nu,t})_{\text{new}}) \leq (1-b^2)\lambda_{\text{new}}^+$ and
 $\lambda_{\text{new}}^- \leq \lambda_{\min}((\Lambda_{a,t_j})_{\text{new}}) \leq \lambda_{\max}((\Lambda_{a,t_j})_{\text{new}}) \leq (1-b^2)\lambda_{\text{new}}^+$

With the above assumptions, clearly, the eigenvalues of $\Lambda_{a,t}$ lie between λ^- and λ^+ and those of $\Lambda_{a,t,\text{new}}$ lie between λ_{new}^- and λ_{new}^+ . Thus the condition numbers of any $\Lambda_{a,t}$ and of $\Lambda_{a,t,\text{new}}$ are bounded by

$$f := \frac{\lambda^+}{\lambda^-} \quad \text{and} \quad g := \frac{\lambda_{\text{new}}^+}{\lambda_{\text{new}}^-}$$

respectively. Define the following quantities for S_t .

Definition 3.4: Let $T_t := \{i : (S_t)_i \neq 0\}$ denote the support of S_t . Define

$$S_{\min} := \min_{t > t_{\text{train}}} \min_{i \in T_t} |(S_t)_i|, \quad \text{and} \quad s := \max_t |T_t|$$

B. Measuring denseness of a matrix and its relation with RIC

Before we can state the denseness assumption, we need to define the denseness coefficient.

Definition 3.5 (denseness coefficient): For a matrix or a vector B , define

$$\kappa_s(B) = \kappa_s(\text{span}(B)) := \max_{|T| \leq s} \|I_T' \text{basis}(B)\|_2 \quad (6)$$

where $\|\cdot\|_2$ is the vector or matrix ℓ_2 -norm.

Clearly, $\kappa_s(B) \leq 1$. First consider an n -length vector B . Then κ_s measures the denseness (non-compressibility) of B . A small value indicates that the entries in B are spread out, i.e. it is a dense vector. A large value indicates that it is compressible (approximately or exactly sparse). The worst case (largest possible value) is $\kappa_s(B) = 1$ which indicates that B is an s -sparse

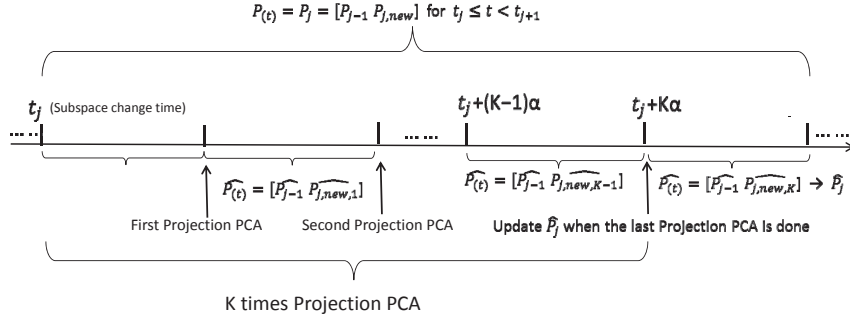


Fig. 1: The K Projection PCA steps.

vector. The best case is $\kappa_s(B) = \sqrt{s/n}$ and this will occur if each entry of B has the same magnitude. Similarly, for an $n \times r$ matrix B , a small κ_s means that most (or all) of its columns are dense vectors.

Remark 3.6: The following facts should be noted about $\kappa_s(\cdot)$.

- 1) For a matrix B of rank r , $\kappa_s(B)$ is a non-decreasing function of s and of r
- 2) A loose bound on $\kappa_s(B)$ obtained using triangle inequality is $\kappa_s(B) \leq s\kappa_1(B)$.

The lemma below relates the denseness coefficient of a basis matrix P to the RIC of $I - PP'$. The proof is in the Appendix.

Lemma 3.7: For an $n \times r$ basis matrix P (i.e P satisfying $P'P = I$),

$$\delta_s(I - PP') = \kappa_s^2(P).$$

In other words, if P is dense enough (small κ_s), then the RIC of $I - PP'$ is small.

In this work, we assume an upper bound on $\kappa_{2s}(P_j)$ for all j , and a tighter upper bound on $\kappa_{2s}(P_{j,new})$, i.e., there exist $\kappa_{2s,*}^+ < 1$ and a $\kappa_{2s,new}^+ < \kappa_{2s,*}^+$ such that

$$\max_j \kappa_{2s}(P_{j-1}) \leq \kappa_{2s,*}^+ \quad (7)$$

$$\max_j \kappa_{2s}(P_{j,new}) \leq \kappa_{2s,new}^+ \quad (8)$$

Additionally, we also assume denseness of another matrix, $D_{j,new,k}$, whose columns span the currently unestimated part of $\text{span}(P_{j,new})$ (see Theorem 4.2).

The denseness coefficient $\kappa_s(B)$ is related to the denseness assumption required by PCP [4]. That work uses $\kappa_1(B)$ to quantify denseness.

IV. THE REPROCS ALGORITHM AND ITS PERFORMANCE GUARANTEE

A. Recursive Projected CS

We summarize the ReProCS algorithm in Algorithm 2 [8]. This calls the projection PCA algorithm in the subspace update step. Given a data matrix \mathcal{D} , a basis matrix P and an integer r , projection-PCA (proj-PCA) applies PCA on $\mathcal{D}_{\text{proj}} := (I - PP')\mathcal{D}$, i.e., it computes the top r eigenvectors (the eigenvectors with the largest r eigenvalues) of $\frac{1}{\alpha_{\mathcal{D}}} \mathcal{D}_{\text{proj}} \mathcal{D}_{\text{proj}}'$. Here $\alpha_{\mathcal{D}}$ is the number of column vectors in \mathcal{D} . This is summarized in Algorithm 1.

If $P = [\cdot]$, then projection-PCA reduces to standard PCA, i.e. it computes the top r eigenvectors of $\frac{1}{\alpha_{\mathcal{D}}} \mathcal{D} \mathcal{D}'$.

Algorithm 1 projection-PCA: $Q \leftarrow \text{proj-PCA}(\mathcal{D}, P, r)$

1) Projection: compute $\mathcal{D}_{\text{proj}} \leftarrow (I - PP')\mathcal{D}$

2) PCA: compute $\frac{1}{\alpha_{\mathcal{D}}} \mathcal{D}_{\text{proj}} \mathcal{D}_{\text{proj}}' \stackrel{EVD}{=} \begin{bmatrix} Q & Q_{\perp} \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda_{\perp} \end{bmatrix} \begin{bmatrix} Q' \\ Q_{\perp}' \end{bmatrix}$ where Q is an $n \times r$ basis matrix and $\alpha_{\mathcal{D}}$ is the number of columns in \mathcal{D} .

Algorithm 2 Recursive Projected CS (ReProCS)

Parameters: algorithm parameters: ξ, ω, α, K , model parameters: t_j, r_0, c

(set as in Theorem 4.2 or as in [8, Section IX-B] when the model is not known)

Input: M_t , *Output:* $\hat{S}_t, \hat{L}_t, \hat{P}_{(t)}$

Initialization: Compute $\hat{P}_0 \leftarrow \text{proj-PCA}([L_1, L_2, \dots, L_{t_{\text{train}}}], [\cdot], r_0)$ and set $\hat{P}_{(t)} \leftarrow \hat{P}_0$.

Let $j \leftarrow 1, k \leftarrow 1$.

For $t > t_{\text{train}}$, do the following:

1) Estimate T_t and S_t via Projected CS:

- a) Nullify most of L_t : compute $\Phi_{(t)} \leftarrow I - \hat{P}_{(t-1)}\hat{P}'_{(t-1)}$, compute $y_t \leftarrow \Phi_{(t)}M_t$
- b) Sparse Recovery: compute $\hat{S}_{t,\text{cs}}$ as the solution of $\min_x \|x\|_1$ s.t. $\|y_t - \Phi_{(t)}x\|_2 \leq \xi$
- c) Support Estimate: compute $\hat{T}_t = \{i : |(\hat{S}_{t,\text{cs}})_i| > \omega\}$
- d) LS Estimate of S_t : compute $(\hat{S}_t)_{\hat{T}_t} = ((\Phi_{(t)})_{\hat{T}_t})^\dagger y_t$, $(\hat{S}_t)_{\hat{T}_t^c} = 0$

2) Estimate L_t : $\hat{L}_t = M_t - \hat{S}_t$.

3) Update $\hat{P}_{(t)}$: K Projection PCA steps (Figure 1).

- a) If $t = t_j + k\alpha - 1$,
 - i) $\hat{P}_{j,\text{new},k} \leftarrow \text{proj-PCA}(\left[\hat{L}_{t_j+(k-1)\alpha}, \dots, \hat{L}_{t_j+k\alpha-1}\right], \hat{P}_{j-1}, c)$.
 - ii) set $\hat{P}_{(t)} \leftarrow [\hat{P}_{j-1} \ \hat{P}_{j,\text{new},k}]$; increment $k \leftarrow k + 1$.
- Else
 - i) set $\hat{P}_{(t)} \leftarrow \hat{P}_{(t-1)}$.
- b) If $t = t_j + K\alpha - 1$, then set $\hat{P}_j \leftarrow [\hat{P}_{j-1} \ \hat{P}_{j,\text{new},K}]$. Increment $j \leftarrow j + 1$. Reset $k \leftarrow 1$.

4) Increment $t \leftarrow t + 1$ and go to step 1.

The key idea of ReProCS is as follows. First, consider a time t when the current basis matrix $P_{(t)} = P_{(t-1)}$ and this has been accurately predicted using past estimates of L_t , i.e. we have $\hat{P}_{(t-1)}$ with $\|(I - \hat{P}_{(t-1)}\hat{P}'_{(t-1)})P_{(t)}\|_2$ small. We project the measurement vector, M_t , into the space perpendicular to $\hat{P}_{(t-1)}$ to get the projected measurement vector $y_t := \Phi_{(t)}M_t$ where $\Phi_{(t)} = I - \hat{P}_{(t-1)}\hat{P}'_{(t-1)}$ (step 1a). Since the $n \times n$ projection matrix, $\Phi_{(t)}$ has rank $n - r_*$ where $r_* = \text{rank}(\hat{P}_{(t-1)})$, therefore y_t has only $n - r_*$ “effective” measurements¹, even though its length is n . Notice that y_t can be rewritten as $y_t = \Phi_{(t)}S_t + \beta_t$ where $\beta_t := \Phi_{(t)}L_t$. Since $\|(I - \hat{P}_{(t-1)}\hat{P}'_{(t-1)})P_{(t-1)}\|_2$ is small, the projection nullifies most of the contribution of L_t and so the projected noise β_t is small. Recovering the n dimensional sparse vector S_t from y_t now becomes a traditional sparse recovery or CS problem in small noise [20], [21], [22], [15], [23], [24]. We use ℓ_1 minimization to recover it (step 1b). If the current basis matrix $P_{(t)}$, and hence its estimate, $\hat{P}_{(t-1)}$, is dense enough, then, by Lemma 3.7, the RIC of $\Phi_{(t)}$ is small enough. Using Theorem 2.5, this ensures that S_t can be accurately recovered from y_t . By thresholding on the recovered S_t , one gets an estimate of its support (step 1c). By computing a least squares (LS) estimate of S_t on the estimated support and setting it to zero everywhere else (step 1d), we can get a more accurate final estimate, \hat{S}_t , as first suggested in [25]. This \hat{S}_t is used to estimate L_t as $\hat{L}_t = M_t - \hat{S}_t$. As we explain in the proof of Lemma 9.1, if S_{\min} is large enough and the support estimation threshold, ω , is chosen appropriately, we can get exact support recovery, i.e. $\hat{T}_t = T_t$. In this case, the error $e_t := \hat{S}_t - S_t = L_t - \hat{L}_t$ has the following simple expression:

$$e_t = I_{T_t}(\Phi_{(t)})_{T_t}^\dagger \beta_t = I_{T_t}[(\Phi_{(t)})'_{T_t}(\Phi_{(t)})_{T_t}]^{-1} I_{T_t}' \Phi_{(t)} L_t \quad (9)$$

The second equality follows because $(\Phi_{(t)})_T' \Phi_{(t)} = (\Phi_{(t)} I_T)' \Phi_{(t)} = I_T' \Phi_{(t)}$ for any set T .

Now consider a time t when $P_{(t)} = P_j = [P_{j-1}, P_{j,\text{new}}]$ and P_{j-1} has been accurately estimated but P_{new} has not been estimated, i.e. consider a $t \in \mathcal{I}_{j,1}$. At this time, $\hat{P}_{(t-1)} = \hat{P}_{j-1}$ and so $\Phi_{(t)} = \Phi_{j,0} := I - \hat{P}_{j-1}\hat{P}'_{j-1}$. Let $r := r_0 + (j-1)c$.

¹i.e. some r_* entries of y_t are linear combinations of the other $n - r_*$ entries

Assume that the delay between change times is large enough so that by $t = t_j$, \hat{P}_{j-1} is an accurate enough estimate of P_{j-1} , i.e. $\|\Phi_{j,0}P_{j-1}\|_2 \leq r\zeta \ll 1$. It is easy to see using Lemma 2.10 that $\kappa_s(\Phi_0 P_{\text{new}}) \leq \kappa_s(P_{\text{new}}) + r\zeta$, i.e. $\Phi_0 P_{\text{new}}$ is dense because P_{new} is dense and because \hat{P}_{j-1} is an accurate estimate of P_{j-1} (which is perpendicular to P_{new}). Moreover, using Lemma 3.7, it can be shown that $\phi_0 := \max_{|T| \leq s} \|[(\Phi_0)'_T(\Phi_0)_T]^{-1}\|_2 \leq \frac{1}{1-\delta_s(\Phi_0)} \leq \frac{1}{1-(\kappa_s(P_{j-1})+r\zeta)^2}$. The error e_t still satisfies (9) although its magnitude is not as small. Using the above facts in (9), we get that

$$\|e_t\|_2 \leq \frac{1}{1 - (\kappa_s(P_{j-1}) + r\zeta)^2} [\kappa_s(P_{\text{new}})\sqrt{c}\gamma_{\text{new}} + r\zeta(\sqrt{r}\gamma_* + \sqrt{c}\gamma_{\text{new}})]$$

If $\sqrt{\zeta} < 1/\gamma_*$, all terms containing ζ can be ignored and we get that the above is approximately upper bounded by $\frac{\kappa_s(P_{\text{new}})}{1-\kappa_s^2(P_{j-1})}\sqrt{c}\gamma_{\text{new}}$. Using the denseness assumption, this quantity is a small constant times $\sqrt{c}\gamma_{\text{new}}$, e.g. with the numbers assumed in Theorem 4.2 we get a bound of $0.18\sqrt{c}\gamma_{\text{new}}$. Since $\gamma_{\text{new}} \ll S_{\min}$ and c is assumed to be small, thus, $\|e_t\|_2 = \|S_t - \hat{S}_t\|_2$ is small compared with $\|S_t\|_2$, i.e. S_t is recovered accurately. With each projection PCA step, as we explain below, the error e_t becomes even smaller.

Since $\hat{L}_t = M_t - \hat{S}_t$ (step 2), e_t also satisfies $e_t = L_t - \hat{L}_t$. Thus, a small e_t means that L_t is also recovered accurately. The estimated \hat{L}_t 's are used to obtain new estimates of $P_{j,\text{new}}$ every α frames for a total of $K\alpha$ frames via a modification of the standard PCA procedure, which we call projection PCA (step 3). In the first projection PCA step, we get the first estimate of $P_{j,\text{new}}$, $\hat{P}_{j,\text{new},1}$. For the next α frame interval, $\hat{P}_{(t-1)} = [\hat{P}_{j-1}, \hat{P}_{j,\text{new},1}]$ and so $\Phi_{(t)} = \Phi_{j,1} = I - \hat{P}_{j-1}\hat{P}_{j-1}' - \hat{P}_{j,\text{new},1}\hat{P}_{j,\text{new},1}'$. Using this in the projected CS step reduces the projection noise, β_t , and hence the reconstruction error, e_t , for this interval, as long as $\gamma_{\text{new},k}$ increases slowly enough. Smaller e_t makes the perturbation seen by the second projection PCA step even smaller, thus resulting in an improved second estimate $\hat{P}_{j,\text{new},2}$. Within K updates (K chosen as given in Theorem 4.2), it can be shown that both $\|e_t\|_2$ and the subspace error drop down to a constant times $\sqrt{\zeta}$. At this time, we update \hat{P}_j as $\hat{P}_j = [\hat{P}_{j-1}, \hat{P}_{j,\text{new},K}]$.

B. Main Result

The following definition is needed for Theorem 4.2.

Definition 4.1: 1) Let $r := r_0 + (J-1)c$.

2) Define $\eta = \max\{\frac{c\gamma_+^2}{\lambda_+}, \frac{c\gamma_{\text{new},k}^2}{\lambda_{\text{new}}^+}, k = 1, 2, \dots, K\}$.

3) Define $K(\zeta) := \left\lceil \frac{\log(0.85c\zeta)}{\log 0.6} \right\rceil$

4) Define $\xi_0(\zeta) := \sqrt{c}\gamma_{\text{new}} + \sqrt{\zeta}(\sqrt{r} + \sqrt{c})$

5) With $K = K(\zeta)$, define

$$\alpha_{\text{add}} := \left\lceil (\log 61KJ + 11 \log n) \frac{8 \cdot 192^2 \min(1.2^{4K}\gamma_{\text{new}}^4, \gamma_*^4)}{\zeta^2(\lambda^-)^2} \right\rceil$$

Theorem 4.2: Consider Algorithm 2. Assume that Assumption 3.1 holds with $b \leq 0.4$. Assume also that the initial subspace estimate is accurate enough, i.e. $\|(I - \hat{P}_0\hat{P}_0')P_0\| \leq r_0\zeta$, with

$$\zeta \leq \min\left(\frac{10^{-4}}{r^2}, \frac{1.5 \times 10^{-4}}{r^2 f}, \frac{1}{r^3 \gamma_*^2}\right) \text{ where } f := \frac{\lambda^+}{\lambda^-}$$

If the following conditions hold:

- 1) the algorithm parameters are set as $\xi = \xi_0(\zeta)$, $7\xi \leq \omega \leq S_{\min} - 7\xi$, $K = K(\zeta)$, $\alpha \geq \alpha_{\text{add}}(\zeta) \geq 100$,
- 2) slow subspace change holds: $t_{j+1} - t_j \geq K\alpha$; (5) holds with $v = 1.2$; and $14\xi \leq S_{\min}$,
- 3) denseness holds: $\max_j \kappa_{2s}(P_{j-1}) \leq \kappa_{2s,*}^+ = 0.3$ and $\max_j \kappa_{2s}(P_{j,\text{new}}) \leq \kappa_{2s,\text{new}}^+ = 0.15$ where

$$\kappa_s(B) := \max_{|T| \leq s} \|I_T' \text{basis}(B)\|_2$$

is the denseness coefficient introduced in [8],

- 4) matrices $D_{j,\text{new},k} := (I - \hat{P}_{j-1}\hat{P}_{j-1}' - \hat{P}_{j,\text{new},k}\hat{P}_{j,\text{new},k}')P_{j,\text{new}}$ and $Q_{j,\text{new},k} := (I - P_{j,\text{new}}P_{j,\text{new}}')\hat{P}_{j,\text{new},k}$ satisfy

$$\kappa_s(D_{j,\text{new},k}) \leq \kappa_s^+ := 0.15,$$

$$\kappa_{2s}(Q_{j,\text{new},k}) \leq \tilde{\kappa}_{2s}^+ := 0.15,$$

- 5) the condition number of the covariance matrix of $a_{t,\text{new}}$ is bounded, i.e., $g \leq g^+ = \sqrt{2}$,
 6) and $\eta \leq 1.7$,
 then, with probability at least $(1 - n^{-10})$,
 1) at all times, t ,

$$\hat{T}_t = T_t \quad \text{and}$$

$$\|e_t\|_2 = \|L_t - \hat{L}_t\|_2 = \|\hat{S}_t - S_t\|_2 \leq 0.18\sqrt{c}0.72^{k-1}\gamma_{\text{new}} + 1.2\sqrt{\zeta}(\sqrt{r} + 0.023\sqrt{c})$$

- 2) the subspace error $\text{SE}_{(t)} := \|(I - \hat{P}_{(t)}\hat{P}'_{(t)})P_{(t)}\|_2$ satisfies

$$\begin{aligned} \text{SE}_{(t)} &\leq \begin{cases} (r_0 + (j-1)c)\zeta + 0.15c\zeta + 0.6^{k-1} & \text{if } t \in \mathcal{I}_{j,k}, k \leq K \\ (r_0 + jc)\zeta & \text{if } t \in \mathcal{I}_{j,K+1} \end{cases} \\ &\leq \begin{cases} 10^{-2}\sqrt{\zeta} + 0.6^{k-1} & \text{if } t \in \mathcal{I}_{j,k}, k \leq K \\ 10^{-2}\sqrt{\zeta} & \text{if } t \in \mathcal{I}_{j,K+1} \end{cases} \end{aligned}$$

- 3) the error $e_t = \hat{S}_t - S_t = L_t - \hat{L}_t$ satisfies the following at various times

$$\begin{aligned} \|e_t\|_2 &\leq \begin{cases} 0.18\sqrt{c}0.72^{k-1}\gamma_{\text{new}} + 1.2(\sqrt{r} + 0.023\sqrt{c})(r_0 + (j-1)c)\zeta\gamma_* & \text{if } t \in \mathcal{I}_{j,k}, k = 1, 2 \dots K \\ 1.2(r_0 + jc)\zeta\sqrt{r}\gamma_* & \text{if } t \in \mathcal{I}_{j,K+1} \end{cases} \\ &\leq \begin{cases} 0.18\sqrt{c}0.72^{k-1}\gamma_{\text{new}} + 1.2(\sqrt{r} + 0.023\sqrt{c})\sqrt{\zeta} & \text{if } t \in \mathcal{I}_{j,k}, k = 1, 2 \dots K \\ 1.2\sqrt{r}\sqrt{\zeta} & \text{if } t \in \mathcal{I}_{j,K+1} \end{cases} \end{aligned}$$

Proof: We give a brief proof outline in Sec IV-D. The full proof is given in Sec VI.

C. Discussion

The above result says the following. Consider Algorithm 2. Assume that the initial subspace error is small enough. If the algorithm parameters are appropriately set, if slow subspace change holds, if the subspaces are dense, if the condition number of $\text{Cov}[a_{t,\text{new}}]$ is small enough, and if the currently unestimated part of the newly added subspace is dense enough (this is an assumption on the algorithm estimates), then, w.h.p., we will get exact support recovery at all times. Moreover, the sparse recovery error will always be bounded by $0.18\sqrt{c}\gamma_{\text{new}}$ plus a constant times $\sqrt{\zeta}$. Since ζ is very small, $\gamma_{\text{new}} \ll S_{\min}$, and c is also small, the normalized reconstruction error for recovering S_t will be small at all times. In the second conclusion, we bound the subspace estimation error, $\text{SE}_{(t)}$. When a subspace change occurs, this error is initially bounded by one. The above result shows that, w.h.p., with each projection PCA step, this error decays exponentially and falls below $0.01\sqrt{\zeta}$ within K projection PCA steps. The third conclusion shows that, with each projection PCA step, w.h.p., the sparse recovery error as well as the error in recovering L_t also decay in a similar fashion.

The above result allows the a_t 's, and hence the L_t 's, to be correlated over time; it models the correlation using an AR model which is a frequently used practical model. Even with this more general model as long as the AR parameter, $b \leq 0.4$, we are able to get almost exactly the same result as that of Qiu et al [8, Theorem 4.1]. The α needed is a little larger. Also, the only extra assumption needed is a small enough upper bound on η which is the ratio of the maximum magnitude entry of any ν_t to the maximum variance. This is true for many types of probability distributions. For example if the i^{th} entry of ν_t is $\pm q_i$ with equal probability independent of all others then $\eta = 1$. If each entry is zero mean uniform distributed (with different spreads) then $\eta = 3$.

Like [8], we still need a denseness assumption on $D_{\text{new},k}$ and $Q_{\text{new},k}$ both of which are functions of algorithm estimates \hat{P}_{j-1} and $\hat{P}_{j,\text{new},k}$. Because of this, our result cannot be interpreted as a correctness result but only a useful step towards it. As explained in [8], from simulation experiments, this assumption does hold whenever there is *some* support changes every few frames. In future work, we hope to be able to replace it with an assumption on the support change of S_t 's.

Also, like [8], the above result analyzes an algorithm that assumes knowledge of the model parameters c , γ_{new} and the subspace change times t_j . Requiring knowledge of c and γ_{new} is not very restrictive. However it also needs to know the

subspace change times t_j and this is somewhat restrictive. One approach to try to remove this requirement is explained in [8].

As explained in [8], under slow subspace change, it is quite valid to assume that the condition number of the new directions, g , is bounded, in fact if at most one new direction could get added, i.e. if $c = 1$, then we would always have $g = 1$. On the other hand, notice that we do not need any bound on f . This is important and needed to allow $\mathbf{E}[\|L_t\|_2^2]$ to be large (L_t is the large but structured noise in case S_t is the signal of interest) while still allowing slow subspace change (needs small γ_{new} and hence small $\lambda^- \leq \gamma_{\text{new}}$). Notice that $\mathbf{E}[\|L_t\|_2^2] \leq r\lambda^+$.

D. Proof Outline

The first step in the proof is to analyze the projected sparse recovery step and show exact support recovery conditioned on the fact that the subspace has been accurately recovered in the previous projection-PCA interval. Exact support recovery along with the LS step allow us to get an exact expression for the recovery error in estimating S_t and hence also for that of L_t . This exact expression is the key to being able to analyze the subspace recovery.

For subspace recovery, the first step involves bounding the subspace recovery error in terms of sub-matrices of the true matrix, $\sum_t \Phi_{(t)} \hat{L}_t \hat{L}_t' \Phi_{(t)}'$, and the perturbation in it, $\sum_t \Phi_{(t)} (\hat{L}_t \hat{L}_t' - L_t L_t') \Phi_{(t)}'$, using the famous $\sin \theta$ theorem [18]. This result bounds the error in the eigenvectors of a matrix perturbed by a Hermitian perturbation. The second step involves obtaining high probability bounds on each of the terms in this bound using the matrix Azuma inequality [26]. The third step involves using the assumptions of the theorem to show that this bound decays roughly exponentially with k and finally falls below $c\zeta$ within K proj-PCA steps.

The most important difference w.r.t. the result of [8] is the following. Define the random variable $X_{j,k} := [\nu_0, \nu_1, \dots, \nu_{t_j+k\alpha-1}]$. In the second step, we need to bound the minimum or maximum singular values of sub-matrices of terms of the form $\sum_t f_1(X_{j,k-1}) a_t a_t' f_2(X_{j,k-1})$ where $f_1(\cdot), f_2(\cdot)$ are functions of the random variable $X_{j,k-1}$. In Qiu et al [8], one could use a simple corollary of the matrix Hoeffding inequality [26] to do this because there, the terms of this summation were conditionally independent given $X_{j,k-1}$. However, here they are not. We instead need to first use the AR model to rewrite things in terms of sub-matrices of $\sum_t f_1(X_{j,k-1}) \nu_t f_3(\nu_0, \nu_1, \dots, \nu_{t-1})' f_2(X_{j,k-1})$. Notice that even now, the terms of this summation are not conditionally independent given $X_{j,k-1}$. However, conditioned on $X_{j,k-1}$, this term is now in a form for which the matrix Azuma inequality [26] can be used.

Notice that the ReProCS algorithm does not need knowledge of b . If b were known, one could modify the algorithm to do proj-PCA on $(\hat{L}_t - b\hat{L}_{t-1})$'s. With this one could use the exact same proof strategy as in [8].

V. SIMULATION

In this section, we compare ReProCS with PCP using simulated data that satisfies the assumed signal model. The data was generated as explained in [8, Section X-C] except that here we generate correlated a_t 's using $a_t = ba_{t-1} + \nu_t$ with $b = 0.5$ and with $\nu_{t,*}$ being uniformly distributed between $[-\gamma_*, \gamma_*]$ and $\nu_{t,\text{new}}$ uniformly distributed between $[-\gamma_{\text{new},k}, \gamma_{\text{new},k}]$. Also we set $t_{\text{train}} = 40, S_{\text{min}} = 2, S_{\text{max}} = 3, s = 7, r_0 = 12, n = 200, J = 2, v = 1.1$ and the support T_t was constant for every set of 50 frames and then changed by 1 index. Other parameters were the same as those in [8, Section X-C]. By running 100 Monte Carlo simulations, we got the result shown in Figure 2.

VI. PROOF OF THEOREM 4.2

Here we first list some definitions used in the proof.

Definition 6.1: We define the noise seen by the sparse recovery step at time t as

$$\beta_t := \|(I - \hat{P}_{(t-1)} \hat{P}_{(t-1)}') L_t\|_2.$$

Definition 6.2: We define the subspace estimation errors as follows. Recall that $\hat{P}_{j,\text{new},0} = [\cdot]$ (empty matrix).

$$\begin{aligned} \text{SE}_{(t)} &:= \|(I - \hat{P}_{(t)} \hat{P}_{(t)}') P_{(t)}\|_2, \\ \zeta_{j,*} &:= \|(I - \hat{P}_{j-1} \hat{P}_{j-1}') P_{j-1}\|_2 \end{aligned}$$

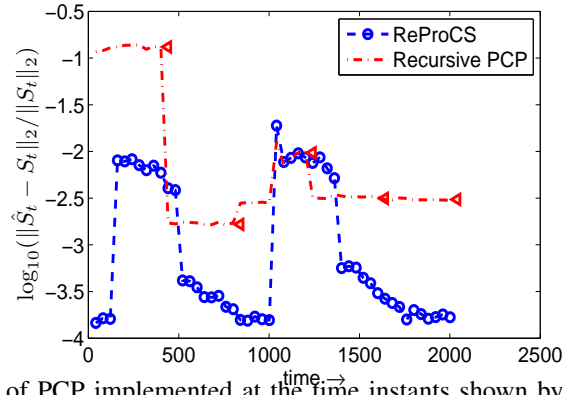


Fig. 2: Comparing recovery error of PCP implemented at the time instants shown by the triangles and of ReProCS.

$$\zeta_{j,k} := \|(I - \hat{P}_{j-1}\hat{P}'_{j-1} - \hat{P}_{j,\text{new},k}\hat{P}'_{j,\text{new},k})P_{j,\text{new}}\|_2$$

Remark 6.3: Recall from the model given in Sec III-A and from Algorithm 2 that

- 1) $\hat{P}_{j,\text{new},k}$ is orthogonal to \hat{P}_{j-1} , i.e. $\hat{P}'_{j,\text{new},k}\hat{P}_{j-1} = 0$
- 2) $\hat{P}_{j-1} := [\hat{P}_0, \hat{P}_{1,\text{new}}, \dots, \hat{P}_{j-1,\text{new}}, K]$ and $P_{j-1} := [P_0, P_{1,\text{new}}, \dots, P_{j-1,\text{new}}]$
- 3) for $t \in \mathcal{I}_{j,k+1}$, $\hat{P}_{(t)} = [\hat{P}_{j-1}, \hat{P}_{j,\text{new},k}]$ and $P_{(t)} = P_j = [P_{j-1}, P_{j,\text{new}}]$.
- 4) $\Phi_{(t)} := I - \hat{P}_{(t-1)}\hat{P}'_{(t-1)}$

From Definition 6.2 and the above, it is easy to see that

- 1) $\zeta_{j,*} \leq \zeta_{1,*} + \sum_{j'=1}^{j-1} \zeta_{j',K}$
- 2) $\text{SE}_{(t)} \leq \zeta_{j,*} + \zeta_{j,k} \leq \zeta_{1,*} + \sum_{j'=1}^{j-1} \zeta_{j',K} + \zeta_{j,k}$ for $t \in \mathcal{I}_{j,k+1}$.

Definition 6.4: Define the following

- 1) $\Phi_{j,k}$, $\Phi_{j,0}$ and ϕ_k
 - a) $\Phi_{j,k} := I - \hat{P}_{j-1}\hat{P}'_{j-1} - \hat{P}_{j,\text{new},k}\hat{P}'_{j,\text{new},k}$ is the CS matrix for $t \in \mathcal{I}_{j,k+1}$, i.e. $\Phi_{(t)} = \Phi_{j,k}$ for this duration.
 - b) $\Phi_{j,0} := I - \hat{P}_{j-1}\hat{P}'_{j-1}$ is the CS matrix for $t \in \mathcal{I}_{j,1}$, i.e. $\Phi_{(t)} = \Phi_{j,0}$ for this duration. $\Phi_{j,0}$ is also the matrix used in all of the projection PCA steps for $t \in [t_j, t_{j+1} - 1]$.
 - c) $\phi_k := \max_j \max_{T: |T| \leq s} \|((\Phi_{j,k})_T)'(\Phi_{j,k})_T)^{-1}\|_2$. It is easy to see that $\phi_k \leq \frac{1}{1 - \max_j \delta_s(\Phi_{j,k})}$ [15].
- 2) $D_{j,\text{new},k}$, $D_{j,\text{new}}$ and $D_{j,*}$
 - a) $D_{j,\text{new},k} := \Phi_{j,k}P_{j,\text{new}}$. $\text{span}(D_{j,\text{new},k})$ is the unestimated part of the newly added subspace for any $t \in \mathcal{I}_{j,k+1}$.
 - b) $D_{j,\text{new}} := D_{j,\text{new},0} = \Phi_{j,0}P_{j,\text{new}}$. $\text{span}(D_{j,\text{new}})$ is interpreted similarly for any $t \in \mathcal{I}_{j,1}$.
 - c) $D_{j,*,k} := \Phi_{j,k}P_{j-1}$. $\text{span}(D_{j,*,k})$ is the unestimated part of the existing subspace for any $t \in \mathcal{I}_{j,k}$.
 - d) $D_{j,*} := D_{j,*,0} = \Phi_{j,0}P_{j-1}$. $\text{span}(D_{j,*,k})$ is interpreted similarly for any $t \in \mathcal{I}_{j,1}$.
 - e) Notice that $\zeta_{j,0} = \|D_{j,\text{new}}\|_2$, $\zeta_{j,k} = \|D_{j,\text{new},k}\|_2$, $\zeta_{j,*} = \|D_{j,*}\|_2$. Also, clearly, $\|D_{j,*,k}\|_2 \leq \zeta_{j,*}$.

Definition 6.5:

- 1) Let $D_{j,\text{new}} \stackrel{QR}{=} E_{j,\text{new}}R_{j,\text{new}}$ denote its QR decomposition. Here $E_{j,\text{new}}$ is a basis matrix and $R_{j,\text{new}}$ is upper triangular.
- 2) Let $E_{j,\text{new},\perp}$ be a basis matrix for the orthogonal complement of $\text{span}(E_{j,\text{new}}) = \text{span}(D_{j,\text{new}})$. To be precise, $E_{j,\text{new},\perp}$ is a $n \times (n - c_{j,\text{new}})$ basis matrix that satisfies $E'_{j,\text{new},\perp}E_{j,\text{new}} = 0$.
- 3) Using $E_{j,\text{new}}$ and $E_{j,\text{new},\perp}$, define $A_{j,k}$, $A_{j,k,\perp}$, $H_{j,k}$, $H_{j,k,\perp}$ and $B_{j,k}$ as

$$\begin{aligned} A_{j,k} &:= \frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} E_{j,\text{new}}' \Phi_{j,0} L_t L_t' \Phi_{j,0} E_{j,\text{new}} \\ A_{j,k,\perp} &:= \frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} E_{j,\text{new},\perp}' \Phi_{j,0} L_t L_t' \Phi_{j,0} E_{j,\text{new},\perp} \\ H_{j,k} &:= \frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} E_{j,\text{new}}' \Phi_{j,0} (e_t e_t' - L_t e_t' - e_t L_t') \Phi_{j,0} E_{j,\text{new}} \end{aligned}$$

$$H_{j,k,\perp} := \frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} E_{j,\text{new},\perp} \Phi_{j,0} (e_t e_t' - L_t e_t' - e_t L_t') \Phi_{j,0} E_{j,\text{new},\perp}$$

$$B_{j,k} := \frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} E_{j,\text{new},\perp} \Phi_{j,0} \hat{L}_t \hat{L}_t' \Phi_{j,0} E_{j,\text{new}} = \frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} E_{j,\text{new},\perp} \Phi_{j,0} (L_t - e_t)(L_t' - e_t') \Phi_{j,0} E_{j,\text{new}}$$

4) Define

$$\mathcal{A}_{j,k} := \begin{bmatrix} E_{j,\text{new}} & E_{j,\text{new},\perp} \end{bmatrix} \begin{bmatrix} A_{j,k} & 0 \\ 0 & A_{j,k,\perp} \end{bmatrix} \begin{bmatrix} E_{j,\text{new}}' \\ E_{j,\text{new},\perp}' \end{bmatrix}$$

$$\mathcal{H}_{j,k} := \begin{bmatrix} E_{j,\text{new}} & E_{j,\text{new},\perp} \end{bmatrix} \begin{bmatrix} H_{j,k} & B_{j,k}' \\ B_{j,k} & H_{j,k,\perp} \end{bmatrix} \begin{bmatrix} E_{j,\text{new}}' \\ E_{j,\text{new},\perp}' \end{bmatrix}$$

5) From the above, it is easy to see that

$$\mathcal{A}_{j,k} + \mathcal{H}_{j,k} = \frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} \Phi_{j,0} \hat{L}_t \hat{L}_t' \Phi_{j,0}.$$

6) Recall from Algorithm 2 that $\mathcal{A}_{j,k} + \mathcal{H}_{j,k} \stackrel{EVD}{=} \begin{bmatrix} \hat{P}_{j,\text{new},k} & \hat{P}_{j,\text{new},k,\perp} \end{bmatrix} \begin{bmatrix} \Lambda_k & 0 \\ 0 & \Lambda_{k,\perp} \end{bmatrix} \begin{bmatrix} \hat{P}_{j,\text{new},k}' \\ \hat{P}_{j,\text{new},k,\perp}' \end{bmatrix}$ is the EVD of $\mathcal{A}_{j,k} + \mathcal{H}_{j,k}$.

Here $\hat{P}_{j,\text{new},k}$ is a $n \times c_{j,\text{new}}$ basis matrix.

7) Using the above, $\mathcal{A}_{j,k} + \mathcal{H}_{j,k}$ can be decomposed in two ways as follows:

$$\begin{aligned} \mathcal{A}_{j,k} + \mathcal{H}_{j,k} &= \begin{bmatrix} \hat{P}_{j,\text{new},k} & \hat{P}_{j,\text{new},k,\perp} \end{bmatrix} \begin{bmatrix} \Lambda_k & 0 \\ 0 & \Lambda_{k,\perp} \end{bmatrix} \begin{bmatrix} \hat{P}_{j,\text{new},k}' \\ \hat{P}_{j,\text{new},k,\perp}' \end{bmatrix} \\ &= \begin{bmatrix} E_{j,\text{new}} & E_{j,\text{new},\perp} \end{bmatrix} \begin{bmatrix} A_{j,k} + H_{j,k} & B_{j,k}' \\ B_{j,k} & A_{j,k,\perp} + H_{j,k,\perp} \end{bmatrix} \begin{bmatrix} E_{j,\text{new}}' \\ E_{j,\text{new},\perp}' \end{bmatrix} \end{aligned}$$

Remark 6.6: Thus, from the above definition, $\mathcal{H}_{j,k} = \frac{1}{\alpha} [\Phi_0 \sum_t (-L_t e_t' - e_t L_t' + e_t e_t') \Phi_0 + F + F']$ where $F := E_{\text{new},\perp} E_{\text{new},\perp}' \Phi_0 \sum_t L_t L_t' \Phi_0 E_{\text{new}} E_{\text{new}}' = E_{\text{new},\perp} E_{\text{new},\perp}' (D_{*,k-1} a_{t,*}) (D_{*,k-1} a_{t,*} + D_{\text{new},k-1} a_{t,\text{new}})' E_{\text{new}} E_{\text{new}}'$. Since $\mathbf{E}[a_{t,*} a_{t,\text{new}}'] = 0$, $\|\frac{1}{\alpha} F\|_2 \lesssim r^2 \zeta^2 \lambda^+$ w.h.p. Recall \lesssim means (in an informal sense) that the RHS contains the dominant terms in the bound.

Definition 6.7: In the sequel, we let

- 1) $\kappa_{s,*} := \max_j \kappa_s(P_{j-1})$, $\kappa_{s,\text{new}} := \max_j \kappa_s(P_{j,\text{new}})$, $\kappa_{s,k} := \max_j \kappa_s(D_{j,\text{new},k})$, $\tilde{\kappa}_{s,k} := \max_j \kappa_s((I - P_{j,\text{new}} P_{j,\text{new}}') \hat{P}_{j,\text{new},k})$,
- 2) $\kappa_{2s,*}^+ := 0.3$, $\kappa_{2s,\text{new}}^+ := 0.15$, $\kappa_s^+ := 0.15$, $\tilde{\kappa}_{2s}^+ := 0.15$ and $g^+ := \sqrt{2}$ are the upper bounds assumed in Theorem 4.2 on $\max_j \kappa_{2s}(P_j)$, $\max_j \kappa_{2s}(P_{j,\text{new}})$, $\max_j \max_k \kappa_s(D_{j,\text{new},k})$, $\max_j \kappa_{2s}(Q_{j,\text{new},k})$ and g respectively.
- 3) $\phi^+ := 1.1735$ We see later that this is an upperbound on ϕ_k under the assumptions of Theorem 4.2.
- 4) $\gamma_{\text{new},k} := \min(1.2^{k-1} \gamma_{\text{new}}, \gamma_*)$
(recall that the theorem assumes $\max_j \max_{t \in \mathcal{I}_{j,k}} \|a_{t,\text{new}}\|_\infty \leq \gamma_{\text{new},k}$)
- 5) $P_{j,*} := P_{j-1}$ and $\hat{P}_{j,*} := \hat{P}_{j-1}$ (see Remark 6.8).

Remark 6.8: Notice that the subscript j always appears as the first subscript, while k is the last one. At many places in this paper, we remove the subscript j for simplicity. Whenever there is only one subscript, it refers to the value of k , e.g., Φ_0 refers to $\Phi_{j,0}$, $\hat{P}_{\text{new},k}$ refers to $\hat{P}_{j,\text{new},k}$. Also, $P_* := P_{j-1}$ and $\hat{P}_* := \hat{P}_{j-1}$.

Definition 6.9: Define the following:

- 1) $\zeta_*^+ := r\zeta$ (We note that $\zeta_*^+ = (r_0 + (j-1)c)\zeta$ will also work.)
- 2) Define the sequence $\{\zeta_k^+\}_{k=0,1,2,\dots,K}$ recursively as follows

$$\zeta_0^+ := 1$$

$$\zeta_k^+ := \frac{b_{\mathcal{H}} + 0.125c\zeta\lambda^-}{b_{A_k} - b_{A_{k,\perp}} - b_{\mathcal{H}} - 0.25c\zeta\lambda^-} \quad \text{for } k \geq 1, \quad (10)$$

where b_{A_k} , $b_{A_{k,\perp}}$, $b_{\mathcal{H}}$ are defined in (59), (61) and (94) respectively.

As we will see, ζ_*^+ and ζ_k^+ are the high probability upper bounds on $\zeta_{j,*}$ and $\zeta_{j,k}$ (defined in Definition 6.2) under the assumptions of Theorem 4.2.

Definition 6.10: Define the random variable $X_{j,k} := \{\nu_1, \nu_2, \dots, \nu_{t_j+k\alpha-1}\}$.

Recall that the ν_t 's are mutually independent over t .

Definition 6.11: Define the set $\tilde{\Gamma}_{j,k}$ as follows:

$$\begin{aligned}\tilde{\Gamma}_{j,k} &:= \{X_{j,k} : \zeta_{j,k} \leq \zeta_k^+ \text{ and } \hat{T}_t = T_t \text{ and } e_t \text{ satisfies (9) for all } t \in \mathcal{I}_{j,k}\} \\ \tilde{\Gamma}_{j,K+1} &:= \{X_{j+1,0} : \hat{T}_t = T_t \text{ and } e_t \text{ satisfies (9) for all } t \in \mathcal{I}_{j,K+1}\}\end{aligned}$$

Definition 6.12: Recursively define the sets $\Gamma_{j,k}$ as follows:

$$\begin{aligned}\Gamma_{1,0} &:= \{X_{1,0} : \zeta_{1,*} \leq r\zeta \text{ and } \hat{T}_t = T_t \text{ and } e_t \text{ satisfies (9) for all } t \in [t_{\text{train}+1} : t_1 - 1]\} \\ \Gamma_{j,k} &:= \Gamma_{j,k-1} \cap \tilde{\Gamma}_{j,k} \quad k = 1, 2, \dots, K, j = 1, 2, \dots, J \\ \Gamma_{j+1,0} &:= \Gamma_{j,K} \cap \tilde{\Gamma}_{j,K+1} \quad j = 1, 2, \dots, J\end{aligned}$$

A. Main Steps for Theorem 4.2

The proof of Theorem 4.2 essentially follows from two main lemmas, 7.1 and 7.2. Lemma 7.1 gives an exponentially decaying upper bound on ζ_k^+ defined in Definition 6.9. ζ_k^+ will be shown to be a high probability upper bound for ζ_k under the assumptions of the Theorem. Lemma 7.2 says that conditioned on $X_{j,k-1} \in \Gamma_{j,k-1}$, $X_{j,k}$ will be in $\Gamma_{j,k}$ w.h.p.. In words this says that if, during the time interval $\mathcal{I}_{j,k-1}$, the algorithm has worked well (recovered the support of S_t exactly and recovered the background subspace with subspace recovery error below $\zeta_{k-1}^+ + \zeta_*^+$), then it will also work well in $\mathcal{I}_{j,k}$ w.h.p.. The proof of Lemma 7.2 requires two lemmas: one for the projected CS step and one for the projection PCA step of the algorithm. These are lemmas 9.1 and 9.2. The proof Lemma 9.1 follows using Lemmas 7.1, 3.7, 2.10, the CS error bound (Theorem 2.5), and some straightforward steps. The proof of Lemma 9.2 is longer and uses a lemma based on the $\sin \theta$ and Weyl theorems (Theorems 2.7 and 2.8) to get a bound on ζ_k . From here we use the matrix Azuma inequalities (Corollaries 2.15 and 2.14) to bound each of the terms in the bound on ζ_k to finally show that, conditioned on $\Gamma_{j,k-1}^e$ $\zeta_k \leq \zeta_k^+$ w.h.p.. These are Lemmas 10.1 and 10.2.

VII. MAIN LEMMAS AND PROOF OF THEOREM 4.2

Recall that when there is only one subscript, it refers to the value of k (i.e. $\zeta_k = \zeta_{j,k}$).

Lemma 7.1 (Exponential decay of ζ_k^+): Pick ζ as given in Theorem 4.2. Assume that the six conditions of Theorem 4.2 hold. Define the series ζ_k^+ as in Definition 6.9. Then,

- 1) $\zeta_0^+ = 1, \zeta_1^+ = 0.5688, \zeta_2^+ = 0.3568, \zeta_k^+ \leq \zeta_{k-1}^+ \leq 0.3568$ for all $k \geq 3$.
- 2) $\zeta_k^+ \leq 0.6^k + 0.15c\zeta$ for all $k \geq 0$
- 3) $b_{A_k} - b_{A_{k,\perp}} - b_{\mathcal{H}} - 0.25c\zeta\lambda^- > 0$ for all $k \geq 1$.

We will prove this lemma in Section VIII.

Lemma 7.2: Assume that all the conditions of Theorem 4.2 hold. Also assume that $\mathbf{P}(\Gamma_{j,k-1}^e) > 0$. Then

$$\mathbf{P}(\Gamma_{j,k}^e | \Gamma_{j,k-1}^e) \geq p_k(\alpha, \zeta) \geq p_K(\alpha, \zeta) \quad \text{for all } k = 1, 2, \dots, K,$$

where $p_k(\alpha, \zeta)$ is defined in equation (21).

Remark 7.3: Under the assumptions of Theorem 4.2, it is easy to see that the following holds. For any $k = 1, 2 \dots K$, $\Gamma_{j,k}^e$ implies that $\zeta_{j,*} \leq \zeta_*^+$

From the definition of $\Gamma_{j,k}^e$, $\zeta_{j',K} \leq \zeta_K^+$ for all $j' \leq j-1$. By Lemma 7.1 and the definition of K in Definition 4.1, $\zeta_K^+ \leq 0.6^K + 0.15c\zeta \leq c\zeta$ for all $j' \leq j-1$. Using Remark 6.3, $\zeta_{j,*} \leq \zeta_1^+ + \sum_{j'=1}^{j-1} \zeta_{j',K} \leq r_0\zeta + (j-1)c\zeta \leq \zeta_*^+$.

Proof of Theorem 4.2:

The theorem is a direct consequence of Lemmas 7.1, 7.2, and Lemma 2.12.

Notice that $\Gamma_{j,0}^e \supseteq \Gamma_{j,1}^e \supseteq \dots \supseteq \Gamma_{j,K,0}^e \supseteq \Gamma_{j+1,0}^e$. Thus, by Lemma 2.12, $\mathbf{P}(\Gamma_{j+1,0}^e | \Gamma_{j,0}^e) = \mathbf{P}(\Gamma_{j+1,0}^e | \Gamma_{j,K}^e) \prod_{k=1}^K \mathbf{P}(\Gamma_{j,k}^e | \Gamma_{j,k-1}^e)$ and $\mathbf{P}(\Gamma_{J+1,0}^e | \Gamma_{1,0}^e) = \prod_{j=1}^J \mathbf{P}(\Gamma_{j+1,0}^e | \Gamma_{j,0}^e)$.

Using Lemmas 7.2, and the fact that $p_k(\alpha, \zeta) \geq p_K(\alpha, \zeta)$ (see their respective definitions in Lemma 10.2 and equation (21)), we get $\mathbf{P}(\Gamma_{J+1,0}^e | \Gamma_{1,0}^e) \geq p_K(\alpha, \zeta)^{KJ}$. Also, $\mathbf{P}(\Gamma_{1,0}^e) = 1$. This follows by the assumption on \hat{P}_0 and Lemma 9.1. Thus, $\mathbf{P}(\Gamma_{J+1,0}^e) \geq p_K(\alpha, \zeta)^{KJ}$.

Using the definition of α_{add} we get that $\mathbf{P}(\Gamma_{J+1,0}^e) \geq p_K(\alpha, \zeta)^{KJ} \geq 1 - n^{-10}$ whenever $\alpha \geq \alpha_{\text{add}}$.

The event $\Gamma_{J+1,0}^e$ implies that $\hat{T}_t = T_t$ and e_t satisfies (9) for all $t < t_{J+1}$. Using Remarks 6.3 and 7.3, $\Gamma_{J+1,0}^e$ implies that all the bounds on the subspace error hold. Using these, $\|a_{t,\text{new}}\|_2 \leq \sqrt{c}\gamma_{\text{new},k}$, and $\|a_t\|_2 \leq \sqrt{r}\gamma_*$, $\Gamma_{J+1,0}^e$ implies that all the bounds on $\|e_t\|_2$ hold (the bounds are obtained in Lemma 9.1).

Thus, all conclusions of the the result hold w.p. at least $1 - n^{-10}$. \blacksquare

VIII. PROOF OF LEMMA 7.1

Proof: By the high probability bounds on $\lambda_{\min}(A_k)$, (59), $\|A_{k,\perp}\|$, (61), $\|H_k\|$, (94) and Assumption 3.1(3g), we have

$$\begin{aligned}
 b_{A_k} &\geq \tilde{b}_{A_k} = (1 - (\zeta_*^+)^2)(1 - \frac{b^2 - b^{2\alpha+2}}{100(1 - b^2)})\lambda_{\text{new},k}^- - 2\frac{b^2(1 - b^{2\alpha})}{1 - b^2}\zeta_*^+ \sqrt{\frac{r}{c}}\eta\lambda^+ \\
 b_{A_{k,\perp}} &\leq \tilde{b}_{A_{k,\perp}} = (\zeta_*^+)^2\lambda^+ + 2\frac{b^2(1 - b^{2\alpha})}{1 - b^2}(\zeta_*^+)^2\frac{r}{c}\eta\lambda^+ \\
 b_{\mathcal{H}} &\leq \tilde{b}_{\mathcal{H}} = (\phi^+)^2(\zeta_*^+)^2\lambda^+ + (\phi^+)^2(\kappa_s^+\zeta_{k-1}^+)^2\lambda_{\text{new},k}^+ + 2\frac{b^2(1 - b^{2\alpha})}{1 - b^2}\sqrt{\frac{r}{c}}\eta\lambda^+\zeta_*^+\zeta_{k-1}^+\kappa_s^+(\phi^+)^2 + \frac{b^2(1 - b^{2\alpha})}{1 - b^2}\eta\frac{r}{c}\lambda^+(\phi^+\zeta_*^+)^2 \\
 &\quad + \frac{b^2(1 - b^{2\alpha})}{1 - b^2}\eta\lambda_{\text{new},k}^+(\phi^+\kappa_s^+\zeta_{k-1}^+)^2 + 2\phi^+\kappa_s^+\frac{(\zeta_*^+)^2}{\sqrt{1 - (\zeta_*^+)^2}}(\lambda^+ + \frac{b^2(1 - b^{2\alpha})}{1 - b^2}\eta\frac{r}{c}\lambda^+) + \\
 &\quad 2\phi^+\max\{0.15c\zeta, \zeta_{k-1}^+\}\frac{(\kappa_s^+)^2}{\sqrt{1 - (\zeta_*^+)^2}}\left(\lambda_{\text{new},k}^+ + \frac{b^2(1 - b^{2\alpha})}{1 - b^2}\eta\lambda_{\text{new},k}^+\right) + 4\frac{b^2(1 - b^{2\alpha})}{1 - b^2}\eta\frac{r}{c}\lambda^+\zeta_*^+\phi^+\frac{\kappa_s^+}{\sqrt{1 - (\zeta_*^+)^2}} + \\
 &\quad (\zeta_*^+ + \phi^+\zeta_*^+)\left(\zeta_*^+ + \zeta_*^+\phi^+\frac{\kappa_s^+}{\sqrt{1 - (\zeta_*^+)^2}}\right)\left(\lambda^+ + \frac{b^2(1 - b^{2\alpha})}{1 - b^2}\eta\lambda^+\right) + \zeta_{k-1}^+\phi^+\kappa_s^+\left(1 + \phi^+\zeta_{k-1}^+\frac{(\kappa_s^+)^2}{\sqrt{1 - (\zeta_*^+)^2}}\right) \\
 &\quad \left(\lambda_{\text{new},k}^+ + \frac{b^2(1 - b^{2\alpha})}{1 - b^2}\eta\lambda_{\text{new},k}^+\right) + \frac{b^2(1 - b^{2\alpha})}{1 - b^2}\sqrt{\frac{r}{c}}\eta\lambda^+\left((\zeta_*^+ + \phi^+\zeta_*^+)\left(1 + (\kappa_s^+)^2\zeta_{k-1}^+\frac{\phi^+}{\sqrt{1 - (\zeta_*^+)^2}}\right)\right. \\
 &\quad \left. + \phi^+\kappa_s^+\zeta_{k-1}^+\left(\zeta_*^+ + \zeta_*^+\phi^+\frac{\kappa_s^+}{\sqrt{1 - (\zeta_*^+)^2}}\right)\right)
 \end{aligned} \tag{11}$$

In the first inequality, we use assumption that $\alpha \geq 100$. When $k = 1$, $\tilde{b}_{\mathcal{H}}$ is a little different because the reason which will be discussed later.

$$\begin{aligned}
 b_{\mathcal{H}_1} &\leq \tilde{b}_{\mathcal{H}_1} = (\phi_0)^2(\zeta_*^+)^2\lambda^+ + (\phi_0)^2(\kappa_s^+\zeta_{k-1}^+)^2\lambda_{\text{new},k}^+ + 2\frac{b^2(1 - b^{2\alpha})}{1 - b^2}\sqrt{\frac{r}{c}}\eta\lambda^+\zeta_*^+\zeta_{k-1}^+\kappa_s^+(\phi_0)^2 + \frac{b^2(1 - b^{2\alpha})}{1 - b^2}\eta\frac{r}{c}\lambda^+(\phi_0\zeta_*^+)^2 + \\
 &\quad 2\phi_0\kappa_s^+\frac{(\zeta_*^+)^2}{\sqrt{1 - (\zeta_*^+)^2}}(\lambda^+ + \frac{b^2(1 - b^{2\alpha})}{1 - b^2}\eta\frac{r}{c}\lambda^+) + 2\phi_0\max\{0.15c\zeta, \zeta_{k-1}^+\}\frac{(\kappa_s^+)^2}{\sqrt{1 - (\zeta_*^+)^2}}\left(\lambda_{\text{new},k}^+ + \frac{b^2(1 - b^{2\alpha})}{1 - b^2}\eta\lambda_{\text{new},k}^+\right) \\
 &\quad + 4\frac{b^2(1 - b^{2\alpha})}{1 - b^2}\eta\frac{r}{c}\lambda^+\zeta_*^+\phi_0\frac{\kappa_s^+}{\sqrt{1 - (\zeta_*^+)^2}} + (\zeta_*^+ + \phi_0\zeta_*^+)\left(\zeta_*^+ + \zeta_*^+\phi_0\frac{\kappa_s^+}{\sqrt{1 - (\zeta_*^+)^2}}\right)\left(\lambda^+ + \frac{b^2(1 - b^{2\alpha})}{1 - b^2}\eta\lambda^+\right) + \\
 &\quad \zeta_{k-1}^+\phi_0\kappa_s^+\left(1 + \phi_0\zeta_{k-1}^+\frac{(\kappa_s^+)^2}{\sqrt{1 - (\zeta_*^+)^2}}\right)\lambda_{\text{new},k}^+ + \frac{b^2(1 - b^{2\alpha})}{1 - b^2}\sqrt{\frac{r}{c}}\eta\lambda^+\left((\zeta_*^+ + \phi_0\zeta_*^+)\left(1 + (\kappa_s^+)^2\zeta_{k-1}^+\frac{\phi_0}{\sqrt{1 - (\zeta_*^+)^2}}\right)\right. \\
 &\quad \left. + \phi_0\kappa_s^+\zeta_{k-1}^+\left(\zeta_*^+ + \zeta_*^+\phi_0\frac{\kappa_s^+}{\sqrt{1 - (\zeta_*^+)^2}}\right)\right)
 \end{aligned} \tag{12}$$

Conditions 2, 4 of Theorem 4.2 imply that $\kappa_{2s,*} \leq \kappa_{2s,*}^+ = 0.3$, $\kappa_{2s,\text{new}} \leq \kappa_{2s,\text{new}}^+ = 0.15$, $\tilde{\kappa}_{2s,k} \leq \tilde{\kappa}_{2s,k}^+ = 0.15$, $\kappa_{s,k} \leq \kappa_{s,k}^+ = 0.15$ and $g \leq g^+ = \sqrt{2}$. Using Lemma 9.3, this implies that $\phi_k \leq \phi^+ = 1.1735$ ($\phi_0 \leq 1.1111$).

Let

$$f_{inc}(\zeta_{k-1}^+; \zeta_*^+, c\zeta, \zeta_*^+ rf, \zeta_*^+ rf, \kappa_s^+, \phi^+, b, \alpha, \eta) = \frac{\tilde{b}_{\mathcal{H}} + 0.125c\zeta\lambda^-}{\tilde{b}_{A_k} - \tilde{b}_{A_{k,\perp}} - \tilde{b}_{\mathcal{H}} - 0.25c\zeta\lambda^-}$$

As $\frac{b^2(1-b^{2\alpha})}{1-b^2} = \sum_{i=0}^{\alpha-1} b^{2(1+i)}$ is an increasing function of b and α , $f_{inc}(\zeta_{k-1}^+; \zeta_*^+, f, c\zeta, \zeta_*^+ rf, \kappa_s^+, \phi^+, b, \alpha, \eta)$ is an increasing function of all their arguments. Thus, by taking upper bounds on some of the variables, we define

$$\tilde{f}_{inc}(\zeta_{k-1}^+; c\zeta, \phi^+, b) = f_{inc}(\zeta_{k-1}^+; 10^{-4}, c\zeta, 1.5 \times 10^{-4}, 1.5 \times 10^{-4}, 0.15, \phi^+, b, \infty, 1.7)$$

Using Fact 9.5, $\zeta_*^+ \leq 10^{-4}$; $\zeta_*^+ f \leq 1.5 \times 10^{-4}$; $\zeta_*^+ rf \leq 1.5 \times 10^{-4}$ and $c\zeta \leq 10^{-4}$.

1) By definition, $\zeta_0^+ = 1$. For $k = 1$, $\zeta_k^+ = \tilde{f}_{inc}(\zeta_{k-1}^+; c\zeta, \phi^+, b) \leq \tilde{f}_{inc}(1; 10^{-4}, 1.1111, 0.4) = 0.5688$. For $k = 2$, $\zeta_k^+ = \tilde{f}_{inc}(\zeta_{k-1}^+; c\zeta, \phi^+, b) \leq \tilde{f}_{inc}(0.5688; 10^{-4}, 1.1735, 0.4) = 0.3568$. We prove the first claim by induction.

- Base case: For $k = 3$, $\zeta_4^+ = \tilde{f}_{inc}(\zeta_{k-1}^+; c\zeta, \phi^+, b) \leq \tilde{f}_{inc}(0.3568; 10^{-4}, 1.1735, 0.4) < 0.3568$.
- Induction step: Assume that $\zeta_{k-1}^+ \leq \zeta_{k-2}^+$ for $k \geq 4$. Since f_{inc} is an increasing function of its arguments, $\zeta_k^+ = \tilde{f}_{inc}(\zeta_{k-1}^+; c\zeta, \phi^+, b) \leq \tilde{f}_{inc}(\zeta_{k-2}^+; c\zeta, \phi^+, b) = \zeta_{k-1}^+$.

2) For the second claim, when $k = 0, 1, 2$, the result is obvious correct; when $k \geq 3$, notice that

$$\begin{aligned} \zeta_k^+ &= \tilde{f}_{inc}(\zeta_{k-1}^+; c\zeta, \phi^+, b) \\ &\leq \tilde{f}_{inc}(\zeta_{k-1}^+; c\zeta, 1.1735, 0.4) \\ &= \frac{0.058\zeta_{k-1}^+ + 0.4284}{0.9981 - 0.058(\zeta_{k-1}^+)^2 - 0.09885\zeta_{k-1}^+ - 0.2505c\zeta} \cdot \zeta_{k-1}^+ + \frac{0.1257c\zeta + 4.041 \times 10^{-5}}{0.9981 - 0.058(\zeta_{k-1}^+)^2 - 0.09885\zeta_{k-1}^+ - 0.2505c\zeta} \\ &\leq 0.5360\zeta_{k-1}^+ + 0.150c\zeta \\ &\leq 0.6^k + 0.15c\zeta \end{aligned} \tag{13}$$

Thus the claim is correct for $k \geq 0$ by induction.

3) Since $\zeta_k^+ \leq \zeta_0^+ = 1$ and

$$g_{dec}(\zeta_{k-1}^+; \zeta_*^+, c\zeta, \zeta_*^+ rf, \zeta_*^+ rf, \kappa_s^+, \phi^+, b, \alpha, \eta) = \frac{b_{A_k} - b_{A_{k,\perp}} - b_{\mathcal{H}} - 0.25c\zeta\lambda^-}{\lambda_{\text{new},k}^-}$$

is a decreasing function of its variables, $g_{dec} \geq g_{dec}(1; 10^{-4}, 10^{-4}, 1.5 \times 10^{-4}, 1.5 \times 10^{-4}, 0.15, 1.1735, 0.4, \infty, 1.7) > 0$. ■

IX. PROOF OF LEMMA 7.2

The proof of Lemma 7.2 follows from two lemmas. The first is the final conclusion for the projected CS step for $t \in \mathcal{I}_{j,k}$. The second is the final conclusion for one projection PCA (i.e.) for $t \in \mathcal{I}_{j,k}$. We will state the two lemmas first and then proceed to prove them in order.

Lemma 9.1 (Projected Compressed Sensing Lemma): Assume that all conditions of Theorem 4.2 hold.

- 1) For all $t \in \mathcal{I}_{j,k}$, for any $k = 1, 2, \dots, K$, if $X_{j,k-1} \in \Gamma_{j,k-1}$,
 - a) the projection noise β_t satisfies $\|\beta_t\|_2 \leq \zeta_{k-1}^+ \sqrt{c}\gamma_{\text{new},k} + \zeta_*^+ \sqrt{r}\gamma_* \leq \sqrt{c}0.72^{k-1}\gamma_{\text{new}} + \sqrt{\zeta}(\sqrt{r} + 0.15\sqrt{c}) \leq \xi_0$.
 - b) the CS error satisfies $\|\hat{S}_{t,\text{cs}} - S_t\|_2 \leq 7\xi_0$.
 - c) $\hat{T}_t = T_t$
 - d) e_t satisfies (9) and $\|e_t\|_2 \leq \phi^+[\kappa_s^+ \zeta_{k-1}^+ \sqrt{c}\gamma_{\text{new},k} + \zeta_*^+ \sqrt{r}\gamma_*] \leq 0.18\sqrt{c}0.72^{k-1}\gamma_{\text{new}} + 1.2\sqrt{\zeta}(\sqrt{r} + 0.023\sqrt{c})$. Recall that (9) is

$$I_{T_t}(\Phi_{(t)})_{T_t}^\dagger \beta_t = I_{T_t}[(\Phi_{(t)})'_{T_t}(\Phi_{(t)})_{T_t}]^{-1} I_{T_t}' \Phi_{(t)} L_t$$

- 2) For all $k = 1, 2, \dots, K$, $\mathbf{P}(\hat{T}_t = T_t \text{ and } e_t \text{ satisfies (9) for all } t \in \mathcal{I}_{j,k} | \Gamma_{j,k-1}^e) = 1$.

Lemma 9.2 (Subspace Recovery Lemma): Assume that all the conditions of Theorem 4.2 hold. Let $\zeta_*^+ = r\zeta$. Then, for all $k = 1, 2, \dots, K$,

$$\mathbf{P}(\zeta_k \leq \zeta_k^+ | \Gamma_{j,k-1}^e) \geq p_k(\alpha, \zeta)$$

where ζ_k^+ is defined in Definition 6.9 and $p_k(\alpha, \zeta)$ is defined in (21).

Proof of Lemma 7.2: Observe that $\mathbf{P}(\Gamma_{j,k}|\Gamma_{j,k-1}) = \mathbf{P}(\tilde{\Gamma}_{j,k}|\Gamma_{j,k-1})$. The lemma then follows by combining Lemma 9.2 and item 2 of Lemma 9.1. ■

A. Proof of Lemma 9.1

In order to prove Lemma 9.1 we first need a bound on the RIC of the compressed sensing matrix Φ_k .

Lemma 9.3 (Bounding the RIC of Φ_k): Recall that $\zeta_* := \|(I - \hat{P}_* \hat{P}_*')P_*\|_2$. The following hold.

- 1) Suppose that a basis matrix P can be split as $P = [P_1, P_2]$ where P_1 and P_2 are also basis matrices. Then $\kappa_s^2(P) = \max_{T: |T| \leq s} \|I_T' P\|_2^2 \leq \kappa_s^2(P_1) + \kappa_s^2(P_2)$.
- 2) $\kappa_s^2(\hat{P}_*) \leq \kappa_{s,*}^2 + 2\zeta_*$
- 3) $\kappa_s(\hat{P}_{\text{new},k}) \leq \kappa_{s,\text{new}} + \tilde{\kappa}_{s,k}\zeta_k + \zeta_*$
- 4) $\delta_s(\Phi_0) = \kappa_s^2(\hat{P}_*) \leq \kappa_{s,*}^2 + 2\zeta_*$
- 5) $\delta_s(\Phi_k) = \kappa_s^2([\hat{P}_* \hat{P}_{\text{new},k}]) \leq \kappa_s^2(\hat{P}_*) + \kappa_s^2(\hat{P}_{\text{new},k}) \leq \kappa_{s,*}^2 + 2\zeta_* + (\kappa_{s,\text{new}} + \tilde{\kappa}_{s,k}\zeta_k + \zeta_*)^2$ for $k \geq 1$

Proof:

- 1) Since P is a basis matrix, $\kappa_s^2(P) = \max_{|T| \leq s} \|I_T' P\|_2^2$. Also, $\|I_T' P\|_2^2 = \|I_T'[P_1, P_2][P_1, P_2]' I_T\|_2 = \|I_T'(P_1 P_1' + P_2 P_2') I_T\|_2 \leq \|I_T' P_1 P_1' I_T\|_2 + \|I_T' P_2 P_2' I_T\|_2$. Thus, the inequality follows.
- 2) For any set T with $|T| \leq s$, $\|I_T' \hat{P}_*\|_2^2 = \|I_T' \hat{P}_* \hat{P}_*' I_T\|_2 = \|I_T'(\hat{P}_* \hat{P}_*' - P_* P_*' + P_* P_*') I_T\|_2 \leq \|I_T'(\hat{P}_* \hat{P}_*' - P_* P_*') I_T\|_2 + \|I_T' P_* P_*' I_T\|_2 \leq 2\zeta_* + \kappa_{s,*}^2$. The last inequality follows using Lemma 2.10 with $P = P_*$ and $\hat{P} = \hat{P}_*$.
- 3) By Lemma 2.10 with $P = P_*$, $\hat{P} = \hat{P}_*$ and $Q = P_{\text{new}}$, $\|P_{\text{new}}' \hat{P}_*\|_2 \leq \zeta_*$. By Lemma 2.10 with $P = P_{\text{new}}$ and $\hat{P} = \hat{P}_{\text{new},k}$, $\|(I - P_{\text{new}} P_{\text{new}}') \hat{P}_{\text{new},k}\|_2 = \|(I - \hat{P}_{\text{new},k} \hat{P}_{\text{new},k}') P_{\text{new}}\|_2$. For any set T with $|T| \leq s$, $\|I_T' \hat{P}_{\text{new},k}\|_2 \leq \|I_T'(I - P_{\text{new}} P_{\text{new}}') \hat{P}_{\text{new},k}\|_2 + \|I_T' P_{\text{new}} P_{\text{new}}' \hat{P}_{\text{new},k}\|_2 \leq \tilde{\kappa}_{s,k} \|(I - P_{\text{new}} P_{\text{new}}') \hat{P}_{\text{new},k}\|_2 + \|I_T' P_{\text{new}}\|_2 = \tilde{\kappa}_{s,k} \|(I - \hat{P}_{\text{new},k} \hat{P}_{\text{new},k}') P_{\text{new}}\|_2 + \|I_T' P_{\text{new}}\|_2 \leq \tilde{\kappa}_{s,k} \|D_{\text{new},k}\|_2 + \tilde{\kappa}_{s,k} \|\hat{P}_* \hat{P}_*' P_{\text{new}}\|_2 + \|I_T' P_{\text{new}}\|_2 \leq \tilde{\kappa}_{s,k} \zeta_k + \tilde{\kappa}_{s,k} \zeta_* + \kappa_{s,\text{new}} \leq \tilde{\kappa}_{s,k} \zeta_k + \zeta_* + \kappa_{s,\text{new}}$. Taking max over $|T| \leq s$ the claim follows.
- 4) This follows using Lemma 3.7 and the second claim of this lemma.
- 5) This follows using Lemma 3.7 and the first three claims of this lemma. ■

Corollary 9.4: If the conditions of Theorem 4.2 are satisfied, and $X_{j,k-1} \in \Gamma_{j,k-1}$, then

- 1) $\delta_s(\Phi_0) \leq \delta_{2s}(\Phi_0) \leq \kappa_{2s,*}^2 + 2\zeta_*^+ < 0.1$
- 2) $\delta_s(\Phi_{k-1}) \leq \delta_{2s}(\Phi_{k-1}) \leq \kappa_{2s,*}^2 + 2\zeta_*^+ + (\kappa_{2s,\text{new}}^+ + \tilde{\kappa}_{2s,k-1}^+ \zeta_{k-1}^+ + \zeta_*^+)^2 < 0.1479$
- 3) $\phi_{k-1} \leq \frac{1}{1 - \delta_s(\Phi_{k-1})} < \phi^+$

Proof: This follows using Lemma 9.3, the definition of $\Gamma_{j,k-1}$, and the bound on ζ_{k-1}^+ from Lemma 7.1. ■

The following are straightforward bounds that will be useful for the proof of Lemma 9.1 and later.

Fact 9.5: Under the assumptions of Theorem 4.2:

- 1) $\zeta \gamma_* \leq \frac{\sqrt{\zeta}}{(r_0 + (J-1)c)^{3/2}} \leq \sqrt{\zeta}$
- 2) $\zeta_*^+ \leq \frac{10^{-4}}{(r_0 + (J-1)c)} \leq 10^{-4}$
- 3) $\zeta_*^+ \gamma_*^2 \leq \frac{1}{(r_0 + (J-1)c)^2} \leq 1$
- 4) $\zeta_*^+ \gamma_* \leq \frac{\sqrt{\zeta}}{\sqrt{r_0 + (J-1)c}} \leq \sqrt{\zeta}$
- 5) $\zeta_*^+ f \leq \frac{1.5 \times 10^{-4}}{r_0 + (J-1)c} \leq 1.5 \times 10^{-4}$
- 6) $\zeta_{k-1}^+ \leq 0.6^{k-1} + 0.15c\zeta$ (from Lemma 7.1)
- 7) $\zeta_{k-1}^+ \gamma_{\text{new},k} \leq (0.6 \cdot 1.2)^{k-1} \gamma_{\text{new}} + 0.15c\zeta \gamma_* \leq 0.72^{k-1} \gamma_{\text{new}} + \frac{0.15\sqrt{\zeta}}{\sqrt{r_0 + (J-1)c}} \leq 0.72^{k-1} \gamma_{\text{new}} + 0.15\sqrt{\zeta}$
- 8) $\zeta_{k-1}^+ \gamma_{\text{new},k}^2 \leq (0.6 \cdot 1.2^2)^{k-1} \gamma_{\text{new}}^2 + 0.15c\zeta \gamma_*^2 \leq 0.864^{k-1} \gamma_{\text{new}}^2 + \frac{0.15}{(r_0 + (J-1)c)^2} \leq 0.864^{k-1} \gamma_{\text{new}}^2 + 0.15$

Proof of Lemma 9.1: Recall that $X_{j,k-1} \in \Gamma_{j,k-1}$ implies that $\zeta_* \leq \zeta_*^+$ and $\zeta_{k-1} \leq \zeta_{k-1}^+$.

- 1) a) For $t \in \mathcal{I}_{j,k}$, $\beta_t := (I - \hat{P}_{(t-1)}\hat{P}'_{(t-1)})L_t = D_{*,k-1}a_{t,*} + D_{\text{new},k-1}a_{t,\text{new}}$. Thus, using Fact 9.5

$$\begin{aligned}\|\beta_t\|_2 &\leq \zeta_*\sqrt{r}\gamma_* + \zeta_{k-1}\sqrt{c}\gamma_{\text{new},k} \\ &\leq \sqrt{\zeta}\sqrt{r} + (0.72^{k-1}\gamma_{\text{new}} + 0.15\sqrt{\zeta})\sqrt{c} \\ &= \sqrt{c}0.72^{k-1}\gamma_{\text{new}} + \sqrt{\zeta}(\sqrt{r} + 0.15\sqrt{c}) \leq \xi_0.\end{aligned}$$

- b) By Corollary 9.4, $\delta_{2s}(\Phi_{k-1}) < 0.15 < \sqrt{2} - 1$. Given $|T_t| \leq s$, $\|\beta_t\|_2 \leq \xi_0 = \xi$, by Theorem 2.5, the CS error satisfies

$$\|\hat{S}_{t,\text{cs}} - S_t\|_2 \leq \frac{4\sqrt{1 + \delta_{2s}(\Phi_{k-1})}}{1 - (\sqrt{2} + 1)\delta_{2s}(\Phi_{k-1})}\xi_0 < 7\xi_0.$$

- c) Using the above and the definition of ρ , $\|\hat{S}_{t,\text{cs}} - S_t\|_\infty \leq 7\rho\xi_0$. Since $\min_{i \in T_t} |(S_t)_i| \geq S_{\min}$ and $(S_t)_{T_t^c} = 0$, $\min_{i \in T_t} |(\hat{S}_{t,\text{cs}})_i| \geq S_{\min} - 7\rho\xi_0$ and $\min_{i \in T_t^c} |(\hat{S}_{t,\text{cs}})_i| \leq 7\rho\xi_0$. If $\omega < S_{\min} - 7\rho\xi_0$, then $\hat{T}_t \supseteq T_t$. On the other hand, if $\omega > 7\rho\xi_0$, then $\hat{T}_t \subseteq T_t$. Since $S_{\min} > 14\rho\xi_0$ (condition 2 of the theorem) and ω satisfies $7\rho\xi_0 \leq \omega \leq S_{\min} - 7\rho\xi_0$ (condition 1 of the theorem), then the support of S_t is exactly recovered, i.e. $\hat{T}_t = T_t$.
- d) Given $\hat{T}_t = T_t$, the LS estimate of S_t satisfies $(\hat{S}_t)_{T_t} = [(\Phi_{k-1})_{T_t}]^\dagger y_t = [(\Phi_{k-1})_{T_t}]^\dagger (\Phi_{k-1}S_t + \Phi_{k-1}L_t)$ and $(\hat{S}_t)_{T_t^c} = 0$ for $t \in \mathcal{I}_{j,k}$. Also, $(\Phi_{k-1})_{T_t}'\Phi_{k-1} = I_{T_t}'\Phi_{k-1}$ (this follows since $(\Phi_{k-1})_{T_t} = \Phi_{k-1}I_{T_t}$ and $\Phi_{k-1}'\Phi_{k-1} = \Phi_{k-1}$). Using this, the LS error $e_t := \hat{S}_t - S_t$ satisfies (9). Thus, using Fact 9.5 and condition 2 of the theorem,

$$\begin{aligned}\|e_t\|_2 &\leq \phi^+(\zeta_*^+\sqrt{r}\gamma_* + \kappa_{s,k-1}\zeta_{k-1}^+\sqrt{c}\gamma_{\text{new},k}) \\ &\leq 1.2\left(\sqrt{r}\sqrt{\zeta} + \sqrt{c}0.15(0.72)^{k-1} + \sqrt{c}0.023\sqrt{\zeta}\right) \\ &= 0.18\sqrt{c}0.72^{k-1}\gamma_{\text{new}} + 1.2\sqrt{\zeta}(\sqrt{r} + 0.023\sqrt{c}).\end{aligned}$$

- 2) The second claim is just a restatement of the first. ■

X. PROOF OF LEMMA 9.2

The proof of Lemma 9.2 will use the next two lemmas (10.1, and 10.2).

Lemma 10.1: If $\lambda_{\min}(A_k) - \|A_{k,\perp}\|_2 - \|\mathcal{H}_k\|_2 > 0$, then

$$\zeta_k \leq \frac{\|\mathcal{R}_k\|_2}{\lambda_{\min}(A_k) - \|A_{k,\perp}\|_2 - \|\mathcal{H}_k\|_2} \leq \frac{\|\mathcal{H}_k\|_2}{\lambda_{\min}(A_k) - \|A_{k,\perp}\|_2 - \|\mathcal{H}_k\|_2} \quad (14)$$

where $\mathcal{R}_k := \mathcal{H}_k E_{\text{new}}$ and A_k , $A_{k,\perp}$, \mathcal{H}_k are defined in Definition 6.5.

Proof: Since $\lambda_{\min}(A_k) - \|A_{k,\perp}\|_2 - \|\mathcal{H}_k\|_2 > 0$, so $\lambda_{\min}(A_k) > \|A_{k,\perp}\|_2$. Since A_k is of size $c_{\text{new}} \times c_{\text{new}}$ and $\lambda_{\min}(A_k) > \|A_{k,\perp}\|_2$, $\lambda_{c_{\text{new}}+1}(A_k) = \|A_{k,\perp}\|_2$. By definition of EVD, and since Λ_k is a $c_{\text{new}} \times c_{\text{new}}$ matrix, $\lambda_{\max}(\Lambda_{k,\perp}) = \lambda_{c_{\text{new}}+1}(A_k + \mathcal{H}_k)$. By Weyl's theorem (Theorem 2.8), $\lambda_{c_{\text{new}}+1}(A_k + \mathcal{H}_k) \leq \lambda_{c_{\text{new}}+1}(A_k) + \|\mathcal{H}_k\|_2 = \|A_{k,\perp}\|_2 + \|\mathcal{H}_k\|_2$. Therefore, $\lambda_{\max}(\Lambda_{k,\perp}) \leq \|A_{k,\perp}\|_2 + \|\mathcal{H}_k\|_2$ and hence $\lambda_{\min}(A_k) - \lambda_{\max}(\Lambda_{k,\perp}) \geq \lambda_{\min}(A_k) - \|A_{k,\perp}\|_2 - \|\mathcal{H}_k\|_2 > 0$. Apply the sin θ theorem (Theorem 2.7) with $\lambda_{\min}(A_k) - \lambda_{\max}(\Lambda_{k,\perp}) > 0$, we get

$$\|(I - \hat{P}_{\text{new},k}\hat{P}'_{\text{new},k})E_{\text{new}}\|_2 \leq \frac{\|\mathcal{R}_k\|_2}{\lambda_{\min}(A_k) - \lambda_{\max}(\Lambda_{k,\perp})} \leq \frac{\|\mathcal{H}_k\|_2}{\lambda_{\min}(A_k) - \|A_{k,\perp}\|_2 - \|\mathcal{H}_k\|_2}$$

Since $\zeta_k = \|(I - \hat{P}_{\text{new},k}\hat{P}'_{\text{new},k})D_{\text{new}}\|_2 = \|(I - \hat{P}_{\text{new},k}\hat{P}'_{\text{new},k})E_{\text{new}}R_{\text{new}}\|_2 \leq \|(I - \hat{P}_{\text{new},k}\hat{P}'_{\text{new},k})E_{\text{new}}\|_2$, the result follows. The last inequality follows because $\|R_{\text{new}}\|_2 = \|E'_{\text{new}}D_{\text{new}}\|_2 \leq 1$. ■

Lemma 10.2 (High probability bounds for each of the terms in the ζ_k bound (14)): Assume the conditions of Theorem 4.2 hold. Also assume that $\mathbf{P}(\Gamma_{j,k-1}^e) > 0$ for all $1 \leq k \leq K+1$. Then, for all $1 \leq k \leq K$

- 1)

$$\mathbf{P}\left(\lambda_{\min}(A_k) \geq b_{A_k} - \frac{c\zeta\lambda^-}{12}\right) \geq 1 - p_a(\alpha, \zeta) \quad (15)$$

where $b_{A_k} = (1 - (\zeta_*^+)^2)(1 - \frac{b^2 - b^{2\alpha+2}}{\alpha(1-b^2)})\lambda_{\text{new},k}^- - 2\frac{b^2(1-b^{2\alpha})}{1-b^2}\sqrt{c\gamma_*}\gamma_{\text{new},k}\zeta_*^+$ and

$$\begin{aligned} p_a(\alpha, \zeta) = & cF\left(\alpha, \frac{c\zeta\lambda^-}{96}, \frac{(1-b^{2\alpha})(1-b)}{1+b}c\gamma_{\text{new},k}^2\right) + 2cF\left(\alpha, \frac{c\zeta\lambda^-}{96}, 2\frac{b(1-b)(1-b^{2(\alpha-1)})}{1+b}c\gamma_{\text{new},k}^2\right) \\ & + 2cF\left(\alpha, \frac{c\zeta\lambda^-}{96}, 2\frac{(1-b^\alpha)^2}{1+b}c\gamma_{\text{new},k}^2\right) + 2cF\left(\alpha, \frac{c\zeta\lambda^-}{48}, 4\frac{1+b-2\sqrt{b^{\alpha+1}(1+b-b^{\alpha+1})}}{1+b}\sqrt{c\gamma_*}\gamma_{\text{new},k}\zeta_*^+\right) \\ & + 2cF\left(\alpha, \frac{c\zeta\lambda^-}{96}, 4\frac{b(1-b^{2\alpha})}{1+b}c\gamma_{\text{new},k}\gamma_{\text{new},k-1}\right) + 2cF\left(\alpha, \frac{c\zeta\lambda^-}{48}, 8\frac{b(1-b^{2\alpha})}{1+b}\sqrt{c\gamma_*}\gamma_{\text{new},k}\zeta_*^+\right) \end{aligned} \quad (16)$$

2)

$$\mathbf{P}\left(\|A_{k,\perp}\| \leq b_{A_{k,\perp}} + \frac{c\zeta\lambda^-}{24} \middle| X_{j,k-1}\right) \geq 1 - p_b(\alpha, \zeta) \quad (17)$$

where $b_{A_{k,\perp}} = \lambda^+(\zeta_*^+)^2 + \frac{b^2(1-b^{2\alpha})}{1-b^2}r\gamma_*^2(\zeta_*^+)^2$ and

$$\begin{aligned} p_b(\alpha, \zeta) = & (n-c)F\left(\alpha, \frac{c\zeta\lambda^-}{96}, \frac{(1-b^{2\alpha})(1-b)}{1+b}r\gamma_*^2(\zeta_*^+)^2\right) + 2(n-c)F\left(\alpha, \frac{c\zeta\lambda^-}{96}, 2\frac{b(1-b)(1-b^{2(\alpha-1)})}{(1+b)}r\gamma_*^2(\zeta_*^+)^2\right) \\ & + 2(n-c)F\left(\alpha, \frac{c\zeta\lambda^-}{96}, 2\frac{(1-b^\alpha)^2}{1+b}r\gamma_*^2(\zeta_*^+)^2\right) + 2(n-c)F\left(\alpha, \frac{c\zeta\lambda^-}{96}, 4\frac{b(1-b^{2\alpha})}{1+b}r\gamma_*^2(\zeta_*^+)^2\right) \end{aligned} \quad (18)$$

3)

$$\mathbf{P}\left(\|\mathcal{H}_k\|_2 \leq b_{\mathcal{H}} + \frac{c\zeta\lambda^-}{8} \middle| X_{j,k-1}\right) \geq 1 - p_c(\alpha, \zeta) - p_f(\alpha, \zeta) - \max\{p_d(\alpha, \zeta), p_e(\alpha, \zeta)\} \quad (19)$$

where

$$\begin{aligned} b_{\mathcal{H}} := & (\phi^+)^2((\zeta_*^+)^2 + (\kappa_s^+\zeta_{k-1}^+)^2)\lambda^+ + 2\frac{b^2(1-b^{2\alpha})}{1-b^2}\sqrt{c\gamma_*}\gamma_{\text{new},k}\zeta_*^+\zeta_{k-1}^+\kappa_s^+(\phi^+)^2 + \frac{b^2(1-b^{2\alpha})}{1-b^2}r\gamma_*^2(\phi^+\zeta_*^+)^2 + \\ & \frac{b^2(1-b^{2\alpha})}{1-b^2}c\gamma_{\text{new},k-1}^2(\phi^+\kappa_s^+\zeta_{k-1}^+)^2 + 2\phi^+\kappa_s^+\frac{(\zeta_*^+)^2}{\sqrt{1-(\zeta_*^+)^2}}\left(\lambda^+ + \frac{b^2(1-b^{2\alpha})}{1-b^2}r\gamma_*^2\right) + \\ & 2\phi^+\max\{0.15c\zeta, \zeta_{k-1}^+\}\frac{(\kappa_s^+)^2}{\sqrt{1-(\zeta_*^+)^2}}\left(\lambda^+ + \frac{b^2(1-b^{2\alpha})}{1-b^2}c\gamma_{\text{new},k-1}^2\right) + 4\frac{b^2(1-b^{2\alpha})}{1-b^2}r\gamma_*^2\zeta_*^+\phi^+\frac{\kappa_s^+}{\sqrt{1-(\zeta_*^+)^2}} + \\ & (\zeta_*^+ + \phi^+\zeta_*^+)(\zeta_*^+ + \zeta_*^+\phi^+\frac{\kappa_s^+}{\sqrt{1-(\zeta_*^+)^2}})(\lambda^+ + \frac{b^2(1-b^{2\alpha})}{1-b^2}r\gamma_*^2) + \zeta_{k-1}^+\phi^+\kappa_s^+(1 + \phi^+\zeta_{k-1}^+\frac{(\kappa_s^+)^2}{\sqrt{1-(\zeta_*^+)^2}}) \\ & \left(\lambda^+ + \frac{b^2(1-b^{2\alpha})}{1-b^2}c\gamma_{\text{new},k-1}^2\right) + \frac{b^2(1-b^{2\alpha})}{1-b^2}\sqrt{c\gamma_*}\gamma_{\text{new},k}\left((\zeta_*^+ + \phi^+\zeta_*^+)\left(1 + (\kappa_s^+)^2\zeta_{k-1}^+\frac{\phi^+}{\sqrt{1-(\zeta_*^+)^2}}\right)\right. \\ & \left.+ \phi^+\kappa_s^+\zeta_{k-1}^+\left(\zeta_*^+ + \zeta_*^+\phi^+\frac{\kappa_s^+}{\sqrt{1-(\zeta_*^+)^2}}\right)\right) \end{aligned} \quad (20)$$

Proof of Lemma 9.2: Lemma 9.2 now follows by combining Lemmas 10.1 and 10.2 and defining

$$p_k(\alpha, \zeta) := 1 - p_a(\alpha, \zeta) - p_b(\alpha, \zeta) - p_c(\alpha, \zeta) - p_f(\alpha, \zeta) - \max\{p_d(\alpha, \zeta), p_e(\alpha, \zeta)\}. \quad (21)$$

As above, we will start with some simple facts that will be used to prove Lemma 10.2.

For convenience, we will use $\frac{1}{\alpha} \sum_t$ to denote $\frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}}$

Fact 10.3: Under the assumptions of Theorem 4.2 the following are true.

- 1) The matrices $D_{\text{new}}, R_{\text{new}}, E_{\text{new}}, D_*, D_{\text{new},k-1}, \Phi_{k-1}$ are functions of the r.v. $X_{j,k-1}$. All terms that we bound for the first two claims of the lemma are of the form $\frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} Z_t$ where $Z_t = f_1(X_{j,k-1})Y_t f_2(X_{j,k-1})$, Y_t is a sub-matrix of $a_t a_t'$ and $f_1(\cdot)$ and $f_2(\cdot)$ are functions of $X_{j,k-1}$.
- 2) $X_{j,k-1}$ is independent of any a_t for $t \in \mathcal{I}_{j,k}$, and hence the same is true for the matrices $D_{\text{new}}, R_{\text{new}}, E_{\text{new}}, D_*, D_{\text{new},k-1}, \Phi_{k-1}$. Also, a_t 's for different $t \in \mathcal{I}_{j,k}$ are mutually independent. Thus, conditioned on $X_{j,k-1}$, the Z_t 's defined above

are mutually independent.

- 3) All the terms that we bound for the third claim contain e_t . Using the second claim of Lemma 9.1, conditioned on $X_{j,k-1}$, e_t satisfies (9) w.p. one whenever $X_{j,k-1} \in \Gamma_{j,k-1}$. Conditioned on $X_{j,k-1}$, all these terms are also of the form $\frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} Z_t$ with Z_t as defined above, whenever $X_{j,k-1} \in \Gamma_{j,k-1}$. Thus, conditioned on $X_{j,k-1}$, the Z_t 's for these terms are mutually independent, whenever $X_{j,k-1} \in \Gamma_{j,k-1}$.
- 4) It is easy to see that $\|\Phi_{k-1} P_*\|_2 \leq \zeta_*$, $\zeta_0 = \|D_{\text{new}}\|_2 \leq 1$, $\Phi_0 D_{\text{new}} = \Phi'_0 D_{\text{new}} = D_{\text{new}}$, $\|R_{\text{new}}\| \leq 1$, $\|(R_{\text{new}})^{-1}\| \leq 1/\sqrt{1-\zeta_*^2}$, $E_{\text{new},\perp}' D_{\text{new}} = 0$, and $\|E_{\text{new}}' \Phi_0 e_t\| = \|(R'_{\text{new}})^{-1} D'_{\text{new}} \Phi_0 e_t\| = \|(R_{\text{new}})^{-1} D'_{\text{new}} e_t\| \leq \|(R'_{\text{new}})^{-1} D'_{\text{new}} I_{T_t}\| \|e_t\| \leq \frac{\kappa_s^+}{\sqrt{1-\zeta_*^2}} \|e_t\|$. The bounds on $\|R_{\text{new}}\|$ and $\|(R_{\text{new}})^{-1}\|$ follow using Lemma 2.10 and the fact that $\sigma_i(R_{\text{new}}) = \sigma_i(D_{\text{new}})$.
- 5) $X_{j,k-1} \in \Gamma_{j,k-1}$ implies that
 - a) $\zeta_* \leq \zeta_*^+$ (see Remark 7.3)
 - b) $\zeta_{k-1} \leq \zeta_{k-1}^+ \leq 0.6^{k-1} + 0.15c\zeta$ (This follows by the definition of $\Gamma_{j,k-1}$ and Lemma 7.1.)
- 6) Item 5 implies that
 - a) $\lambda_{\min}(R_{\text{new}} R_{\text{new}}') \geq 1 - (\zeta_*^+)^2$. This follows from Lemma 2.10 and the fact that $\sigma_{\min}(R_{\text{new}}) = \sigma_{\min}(D_{\text{new}})$.
 - b) $\|I_{T_t}' \Phi_{k-1} P_*\|_2 \leq \|\Phi_{k-1} P_*\|_2 \leq \zeta_* \leq \zeta_*^+$, $\|I_{T_t}' D_{\text{new},k-1}\|_2 \leq \kappa_{s,k-1} \zeta_{k-1} \leq \kappa_s^+ \zeta_{k-1}^+$.
- 7) By Weyl's theorem (Theorem 2.8), for a sequence of matrices B_t , $\lambda_{\min}(\sum_t B_t) \geq \sum_t \lambda_{\min}(B_t)$ and $\lambda_{\max}(\sum_t B_t) \leq \sum_t \lambda_{\max}(B_t)$.

Proof of Lemma 10.2: In this proof, we frequently refer to items from Sec. 10.3 and the following bounds (22).

$$\begin{aligned}
 \|E'_{\text{new}} \Phi_0\|_2 &\leq 1 \\
 \|E'_{\text{new}} \Phi_0 I_{T_t}\|_2 &= \|(R'_{\text{new}})^{-1} D'_{\text{new}} \Phi_0 I_{T_t}\|_2 = \|(R'_{\text{new}})^{-1} D'_{\text{new}} I_{T_t}\|_2 \leq \frac{\zeta_0 \kappa_s^+}{\sqrt{1-\zeta_*^2}} \leq \frac{\kappa_s^+}{\sqrt{1-(\zeta_*^+)^2}} \\
 \|\Phi_0 P_*\|_2 &= \zeta_* \leq \zeta_*^+ \\
 \|\Phi_{k-1} P_*\|_2 &\leq \|\Phi_0 P_*\|_2 \leq \zeta_*^+ \\
 \|I'_{T_t} D_{\text{new},k-1}\| &\leq \kappa_s^+ \zeta_{k-1}^+
 \end{aligned} \tag{22}$$

A. Bounds for $\sum_{t \in \mathcal{I}_{j,k}} a_{t,\text{new}} a'_{t,\text{new}}, \sum_{t \in \mathcal{I}_{j,k}} a_{t,\text{new}} a'_{t,*}, \sum_{t \in \mathcal{I}_{j,k}} a_{t,*} a'_{t,*}$

For calculation simplicity, let $\{d_t\}_{0 \leq t \leq \alpha-1}$ denote $\{a_{t,\text{new}}\}_{t \in \mathcal{I}_{j,k}}$, $\{\mu_t\}_{0 \leq t \leq \alpha-1}$ denote $\{\nu_{t,\text{new}}\}_{t \in \mathcal{I}_{j,k}}$, $\{m_t\}_{0 \leq t \leq \alpha-1}$ denote $\{a_{t,*}\}_{t \in \mathcal{I}_{j,k}}$, and $\{\omega_t\}_{0 \leq t \leq \alpha-1}$ denote $\{\nu_{t,*}\}_{t \in \mathcal{I}_{j,k}}$.

$$\begin{aligned}
 &\sum_{t=0}^{\alpha-1} d_t d'_t \\
 &= \sum_{t=0}^{\alpha-1} (b d_{t-1} + \mu_t)(b d_{t-1} + \mu_t)' \\
 &= \sum_{t=0}^{\alpha-1} (b^{t+1} d_{-1} + \sum_{i=0}^t b^{t-i} \mu_i)(b^{t+1} d_{-1} + \sum_{i=0}^t b^{t-i} \mu_i)' \\
 &= \sum_{t=0}^{\alpha-1} (b^{2t} \sum_{i=0}^t b^{-2i} \mu_i \mu'_i) + \sum_{t=0}^{\alpha-1} (b^{2t} \sum_{0 \leq i_1, i_2 \leq t, i_1 \neq i_2} b^{-i_1-i_2} \mu_{i_1} \mu'_{i_2}) + \sum_{t=0}^{\alpha-1} \sum_{i=0}^t b^{2t-i+1} d_{-1} \mu'_i + \\
 &\quad \sum_{t=0}^{\alpha-1} \sum_{i=0}^t b^{2t-i+1} \mu_i d'_{-1} + \sum_{t=0}^{\alpha-1} b^{2(t+1)} d_{-1} d'_{-1} \\
 &= \sum_{i=0}^{\alpha-1} \sum_{t=i}^{\alpha-1} (b^{2t} b^{-2i} \mu_i \mu'_i) + \sum_{t=0}^{\alpha-1} b^{2t} (\sum_{i_1=0}^t \sum_{i_2=0}^{i_1-1} b^{-i_1-i_2} \mu_{i_1} \mu'_{i_2} + \sum_{i_1=0}^t \sum_{i_2=i_1+1}^t b^{-i_1-i_2} \mu_{i_1} \mu'_{i_2}) + \\
 &\quad \sum_{i=0}^{\alpha-1} \sum_{t=i}^{\alpha-1} b^{2t-i+1} d_{-1} \mu'_i + \sum_{i=0}^{\alpha-1} \sum_{t=i}^{\alpha-1} b^{2t-i+1} \mu_i d'_{-1} + \frac{b^2(1-b^{2\alpha})}{1-b^2} d_{-1} d'_{-1}
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=0}^{\alpha-1} \frac{(1-b^{2(\alpha-i)})}{1-b^2} \mu_i \mu'_i + \sum_{i_1=0}^{\alpha-1} \sum_{t=i_1}^{\alpha-1} b^{2t} \left(\sum_{i_2=t_j+(k-1)}^{i_1-1} b^{-i_1-i_2} \mu_{i_1} \mu'_{i_2} + \sum_{i_2=i_1+1}^t b^{-i_1-i_2} \mu_{i_1} \mu'_{i_2} \right) + \\
&\quad \sum_{i=0}^{\alpha-1} \frac{b^{i+1}(1-b^{2(\alpha-i)})}{1-b^2} (\mu_i d'_{-1} + d_{-1} \mu'_i) + \frac{b^2(1-b^{2\alpha})}{1-b^2} d_{-1} d'_{-1} \\
&= \sum_{i=0}^{\alpha-1} \frac{(1-b^{2(\alpha-i)})}{1-b^2} \mu_i \mu'_i + \sum_{i_1=0}^{\alpha-1} \frac{(1-b^{2(\alpha-i_1)})}{1-b^2} \sum_{i_2=0}^{i_1-1} b^{i_1-i_2} \mu_{i_1} \mu'_{i_2} + \sum_{i_1=-1}^{\alpha-1} \sum_{i_2=i_1+1}^{\alpha-1} \sum_{t=i_2}^{\alpha-1} b^{2t} b^{-i_1-i_2} \mu_{i_1} \mu'_{i_2} \\
&\quad + \sum_{i=0}^{\alpha-1} \frac{b^{i+1}(1-b^{2(\alpha-i)})}{1-b^2} (\mu_i d'_{-1} + d_{-1} \mu'_i) + \frac{b^2(1-b^{2\alpha})}{1-b^2} d_{-1} d'_{-1} \\
&= \sum_{i=0}^{\alpha-1} \frac{(1-b^{2(\alpha-i)})}{1-b^2} \mu_i \mu'_i + \sum_{i_1=0}^{\alpha-1} \frac{(1-b^{2(\alpha-i_1)})}{1-b^2} \sum_{i_2=0}^{i_1-1} b^{i_1-i_2} \mu_{i_1} \mu'_{i_2} + \sum_{i_1=0}^{\alpha-1} \sum_{i_2=i_1+1}^{\alpha-1} \frac{(1-b^{2(\alpha-i_2)})}{1-b^2} b^{-i_1+i_2} \mu_{i_1} \mu'_{i_2} \\
&\quad + \sum_{i=0}^{\alpha-1} \frac{b^{i+1}(1-b^{2(\alpha-i)})}{1-b^2} (\mu_i d'_{-1} + d_{-1} \mu'_i) + \frac{b^2(1-b^{2\alpha})}{1-b^2} d_{-1} d'_{-1} \\
&= \sum_{i=0}^{\alpha-1} \frac{(1-b^{2(\alpha-i)})}{1-b^2} \mu_i \mu'_i + \sum_{i_1=0}^{\alpha-1} \frac{(1-b^{2(\alpha-i_1)})}{1-b^2} \sum_{i_2=0}^{i_1-1} b^{i_1-i_2} \mu_{i_1} \mu'_{i_2} + \sum_{i_1=0}^{\alpha-1} \sum_{i_2=\alpha-i_1}^{\alpha-1} \frac{(1-b^{2(\alpha-i_2)})}{1-b^2} b^{i_1+i_2-\alpha+1} \mu_{\alpha-i_1-1} \mu'_{i_2} \\
&\quad + \sum_{i=0}^{\alpha-1} \frac{b^{i+1}(1-b^{2(\alpha-i)})}{1-b^2} (\mu_i d'_{-1} + d_{-1} \mu'_i) + \frac{b^2(1-b^{2\alpha})}{1-b^2} d_{-1} d'_{-1}
\end{aligned}$$

Let

$$\begin{aligned}
Z_{1,i} &= \frac{(1-b^{2(\alpha-i)})}{1-b^2} \mu_i \mu'_i, \\
Z_{2,i} &= \sum_{i_2=0}^{i-1} \frac{(1-b^{2(\alpha-i)})}{1-b^2} b^{i-i_2} \mu_i \mu'_{i_2}, i \geq 1 \\
Z_{3,i} &= \sum_{i_2=\alpha-i}^{\alpha-1} \frac{(1-b^{2(\alpha-i_2)})}{1-b^2} b^{i+i_2-\alpha+1} \mu_{\alpha-i-1} \mu'_{i_2}, \\
Z_{4,i} &= \frac{b^{i+1}(1-b^{2(\alpha-i)})}{1-b^2} \mu_i d'_{-1}
\end{aligned} \tag{23}$$

$$Z_{3,i} = \sum_{i_2=\alpha-i}^{\alpha-1} \frac{(1-b^{2(\alpha-i_2)})}{1-b^2} b^{i+i_2-\alpha+1} \mu_{\alpha-i-1} \mu'_{i_2} = \mu_{\alpha-i-1} h_i(\mu_{\alpha-i}, \mu_{\alpha-i+1}, \dots, \mu_{\alpha-1}), \text{ thus}$$

$$\begin{aligned}
\mathbf{E}_{i-1}(Z_{3,i}) &= \mathbf{E}(\mu_{\alpha-i-1} h_i(\mu_{\alpha-i}, \mu_{\alpha-i+1}, \dots, \mu_{\alpha-1}) | g_i(\mu_{\alpha-i}, \mu_{\alpha-i+1}, \dots, \mu_{\alpha-1})) \\
&= \mathbf{E}(\mu_{\alpha-i-1}) \mathbf{E}(h_i(\mu_{\alpha-i}, \mu_{\alpha-i+1}, \dots, \mu_{\alpha-1}) | g_i(\mu_{\alpha-i}, \mu_{\alpha-i+1}, \dots, \mu_{\alpha-1})) \\
&= 0
\end{aligned} \tag{24}$$

Then we can split $\{d_t d'_t\}_{0 \leq t \leq \alpha-1}$ into four adapted sequences, i.e., $\{Z_{1,i}\}_{0 \leq i \leq \alpha-1}$, $\{Z_{2,i}\}_{0 \leq i \leq \alpha-1}$, $\{Z_{3,i}\}_{0 \leq i \leq \alpha-1}$ and $\{Z_{4,i}\}_{0 \leq i \leq \alpha-1}$.

According to Assumption 3.1(3),

$$\begin{aligned}
&\lambda_{\min} \left(\mathbf{E}_{i-1} \left(\frac{1}{\alpha} \sum_{i=0}^{\alpha-1} Z_{1,i} | X_{j,k-1} \right) \right) \\
&\geq \frac{1}{\alpha} \sum_{i=0}^{\alpha-1} \frac{(1-b^{2(\alpha-i)})}{1-b^2} \lambda_{\min}(\mathbf{E}(\mu_i \mu'_i)) \\
&\geq \frac{\alpha(1-b^2) - b^2 + b^{2\alpha+2}}{\alpha(1-b^2)} \lambda_{\text{new},k}^- \\
&= \left(1 - \frac{b^2 - b^{2\alpha+2}}{\alpha(1-b^2)} \right) \lambda_{\text{new},k}^-
\end{aligned}$$

and

$$\begin{aligned}
& \lambda_{\max} \left(\mathbf{E}_{i-1} \left(\frac{1}{\alpha} \sum_{i=0}^{\alpha-1} Z_{1,i} | X_{j,k-1} \right) \right) \\
& \leq \frac{1}{\alpha} \sum_{i=0}^{\alpha-1} \frac{(1-b^{2(\alpha-i)})}{1-b^2} \lambda_{\max}(\mathbf{E}(\mu_i \mu'_i)) \\
& \leq \left(\frac{\alpha(1-b^2) - b^2 + b^{2\alpha+2}}{\alpha(1-b^2)} \lambda_{\text{new},k}^+ \right) \\
& \leq \lambda_{\text{new},k}^+
\end{aligned}$$

and by Lemma 2.17,

$$\begin{aligned}
\mathbf{E}_{i-1}(Z_{2,i} | X_{j,k-1}) &= \mathbf{E}_{i-1} \left(\sum_{i_2=0}^{i-1} \frac{(1-b^{2(\alpha-i)})}{1-b^2} b^{i-i_2} \mu_i \mu'_{i_2} \right) \\
&= \mathbf{E} \left(\mu_i \sum_{i_2=0}^{i-1} \frac{(1-b^{2(\alpha-i)})}{1-b^2} b^{i-i_2} \mu'_{i_2} | Z_{2,0}, \dots, Z_{2,i-1} \right) \\
&= \mathbf{E}(\mu_i h(\mu_0, \mu_1, \dots, \mu_{i-1}) | g(\mu_0, \mu_1, \dots, \mu_{i-1})) \\
&= \mathbf{E}(\mu_i) \mathbf{E}(h(\mu_0, \mu_1, \dots, \mu_{i-1}) | g(\mu_0, \mu_1, \dots, \mu_{i-1})) \\
&= 0,
\end{aligned} \tag{25}$$

Similarly, we have $\mathbf{E}_{i-1}(Z_{3,i} | X_{j,k-1}) = 0$, $\mathbf{E}_{i-1}(Z_{4,i} | X_{j,k-1}) = 0$.

$$\begin{aligned}
0 \preceq Z_{1,i} &\preceq \frac{(1-b^{2(\alpha-i)})}{1-b^2} c(1-b)^2 \gamma_{\text{new},k}^2 I \preceq \frac{(1-b^{2\alpha})(1-b)}{1+b} c \gamma_{\text{new},k}^2 I, \\
\|Z_{2,i_1}\| &\leq \sum_{i_2=0}^{i_1-1} \frac{(1-b^{2(\alpha-i_1)})}{1-b^2} b^{i_1-i_2} c(1-b)^2 \gamma_{\text{new},k}^2 \\
&\leq \frac{b(1-b^{2(\alpha-i_1)})(1-b^{i_1})}{1+b} c \gamma_{\text{new},k}^2 \\
&\leq \frac{b(1-b)(1-b^{2(\alpha-1)})}{1+b} c \gamma_{\text{new},k}^2
\end{aligned} \tag{26}$$

$$\begin{aligned}
\|Z_{3,i}\| &\leq \sum_{i_2=\alpha-i}^{\alpha-1} \frac{(1-b^{2(\alpha-i_2)})}{1-b^2} b^{i+i_2-\alpha+1} c(1-b)^2 \gamma_{\text{new},k}^2 \\
&\leq \frac{(1-b^{i+1})^2}{1+b} c \gamma_{\text{new},k}^2 \\
&\leq \frac{(1-b^\alpha)^2}{1+b} c \gamma_{\text{new},k}^2
\end{aligned} \tag{27}$$

$$\begin{aligned}
\|Z_{4,i}\| &\leq \frac{b^{i+1}(1-b^{2(\alpha-i)})}{1-b^2} c(1-b) \gamma_{\text{new},k} \gamma_{\text{new},k-1} \\
&\leq \frac{b(1-b^{2\alpha})}{1+b} c \gamma_{\text{new},k} \gamma_{\text{new},k-1}
\end{aligned} \tag{28}$$

$$0 \preceq \frac{b^2(1-b^{2\alpha})}{1-b^2} d_{-1} d'_{-1} \preceq \frac{b^2(1-b^{2\alpha})}{1-b^2} c \gamma_{\text{new},k-1}^2 I \tag{29}$$

Given $X_{j,k-1}$, $\{Z_{1,i}\}_{i \in \mathcal{I}_{j,k}}$ is an adapted sequence of Hermitian matrices, thus by Corollary 2.15, for all $X_{j,k-1} \in \Gamma_{j,k-1}$, we have

$$\mathbf{P} \left(\lambda_{\min} \left(\frac{1}{\alpha} \sum_{i=0}^{\alpha-1} Z_{1,i} \right) \geq \left(1 - \frac{b^2 - b^{2\alpha+2}}{\alpha(1-b^2)} \right) \lambda_{\text{new},k}^- - \epsilon \middle| X_{j,k-1} \right) \geq 1 - cF \left(\alpha, \epsilon, \frac{(1-b^{2\alpha})(1-b)}{1+b} c\gamma_{\text{new},k}^2 \right) \quad (30)$$

Similarly, given $X_{j,k-1}$, $\{Z_{2,i_1}\}_{i_1 \in \mathcal{I}_{j,k}}$, $\{Z_{3,i_1}\}_{i_1 \in \mathcal{I}_{j,k}}$, $\{Z_{4,i}\}_{i \in \mathcal{I}_{j,k}}$ are adapted sequences of matrices, by Corollary 2.14, for all $X_{j,k-1} \in \Gamma_{j,k-1}$ we have

$$\mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{i_1=0}^{\alpha-1} Z_{2,i_1} \right\| \leq \epsilon \middle| X_{j,k-1} \right) \geq 1 - 2cF \left(\alpha, \epsilon, 2 \frac{b(1-b)(1-b^{2(\alpha-1)})}{1+b} c\gamma_{\text{new},k}^2 \right) \quad (31)$$

$$\mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{i_1=0}^{\alpha-1} Z_{3,i_1} \right\| \leq \epsilon \middle| X_{j,k-1} \right) \geq 1 - 2cF \left(\alpha, \epsilon, 2 \frac{(1-b^\alpha)^2}{1+b} c\gamma_{\text{new},k}^2 \right) \quad (32)$$

$$\mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{i=0}^{\alpha-1} (Z_{4,i} + Z'_{4,i}) \right\| \leq \epsilon \middle| X_{j,k-1} \right) \geq 1 - 2cF \left(\alpha, \epsilon, 4 \frac{b(1-b^{2\alpha})}{1+b} c\gamma_{\text{new},k} \gamma_{\text{new},k-1} \right) \quad (33)$$

Thus, given $X_{j,k-1}$, we have

$$\begin{aligned} & \mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} a_{t,\text{new}} a'_{t,\text{new}} \right\| \leq \lambda_{\text{new},k}^+ + \frac{b^2(1-b^{2\alpha})}{1-b^2} c\gamma_{\text{new},k-1}^2 + \epsilon \middle| X_{j,k-1} \right) \\ &= 1 - \mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} a_{t,\text{new}} a'_{t,\text{new}} \right\| > \lambda_{\text{new},k}^+ + \frac{b^2(1-b^{2\alpha})}{1-b^2} c\gamma_{\text{new},k-1}^2 + \epsilon \middle| X_{j,k-1} \right) \\ &\geq 1 - \mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{i=0}^{\alpha-1} Z_{1,i} \right\| > \lambda_{\text{new},k}^+ + \frac{\epsilon}{4} \middle| X_{j,k-1} \right) - \mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{i_1=0}^{\alpha-1} Z_{2,i_1} \right\| > \frac{\epsilon}{4} \middle| X_{j,k-1} \right) - \mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{i_1=0}^{\alpha-1} Z_{3,i_1} \right\| > \frac{\epsilon}{4} \middle| X_{j,k-1} \right) \\ &\quad - \mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{i_1=0}^{\alpha-1} (Z_{4,i} + Z'_{4,i}) \right\| > \frac{\epsilon}{4} \middle| X_{j,k-1} \right) - \mathbf{P} \left(\left\| \frac{b^2(1-b^{2\alpha})}{1-b^2} d_{-1} d'_{-1} \right\| > \frac{b^2(1-b^{2\alpha})}{1-b^2} c\gamma_{\text{new},k-1}^2 \right) \\ &\geq 1 - cF \left(\alpha, \frac{\epsilon}{4}, \frac{(1-b^{2\alpha})(1-b)}{1+b} c\gamma_{\text{new},k}^2 \right) - 2cF \left(\alpha, \frac{\epsilon}{4}, 2 \frac{b(1-b)(1-b^{2(\alpha-1)})}{1+b} c\gamma_{\text{new},k}^2 \right) - 2cF \left(\alpha, \frac{\epsilon}{4}, 2 \frac{(1-b^\alpha)^2}{1+b} c\gamma_{\text{new},k}^2 \right) \\ &\quad - 2cF \left(\alpha, \frac{\epsilon}{4}, 4 \frac{b(1-b^{2\alpha})}{1+b} c\gamma_{\text{new},k} \gamma_{\text{new},k-1} \right) \end{aligned} \quad (34)$$

$$\begin{aligned} & \mathbf{P} \left(\lambda_{\min} \left(\frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} a_{t,\text{new}} a'_{t,\text{new}} \right) \geq \left(1 - \frac{b^2 - b^{2\alpha+2}}{\alpha(1-b^2)} \right) \lambda_{\text{new},k}^- - \epsilon \middle| X_{j,k-1} \right) \\ &= 1 - \mathbf{P} \left(\lambda_{\min} \left(\frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} a_{t,\text{new}} a'_{t,\text{new}} \right) < \left(1 - \frac{b^2 - b^{2\alpha+2}}{\alpha(1-b^2)} \right) \lambda_{\text{new},k}^- - \epsilon \middle| X_{j,k-1} \right) \\ &\geq 1 - \mathbf{P} \left(\lambda_{\min} \left(\frac{1}{\alpha} \sum_{i=0}^{\alpha-1} Z_{1,i} \right) < \left(1 - \frac{b^2 - b^{2\alpha+2}}{\alpha(1-b^2)} \right) \lambda_{\text{new},k}^- - \frac{\epsilon}{4} \middle| X_{j,k-1} \right) - \mathbf{P} \left(\lambda_{\min} \left(\frac{1}{\alpha} \sum_{i_1=0}^{\alpha-1} Z_{2,i_1} \right) < -\frac{\epsilon}{4} \middle| X_{j,k-1} \right) \\ &\quad - \mathbf{P} \left(\lambda_{\min} \left(\frac{1}{\alpha} \sum_{i_1=0}^{\alpha-1} Z_{3,i_1} \right) < -\frac{\epsilon}{4} \middle| X_{j,k-1} \right) - \mathbf{P} \left(\lambda_{\min} \left(\frac{1}{\alpha} \sum_{i_1=0}^{\alpha-1} (Z_{4,i} + Z'_{4,i}) \right) < -\frac{\epsilon}{4} \middle| X_{j,k-1} \right) \\ &\quad - \mathbf{P} \left(\lambda_{\min} \left(\frac{b^2(1-b^{2\alpha})}{1-b^2} d_{-1} d'_{-1} \right) < 0 \right) \\ &\geq 1 - cF \left(\alpha, \frac{\epsilon}{4}, \frac{(1-b^{2\alpha})(1-b)}{1+b} c\gamma_{\text{new},k}^2 \right) - 2cF \left(\alpha, \frac{\epsilon}{4}, 2 \frac{b(1-b)(1-b^{2(\alpha-1)})}{1+b} c\gamma_{\text{new},k}^2 \right) - 2cF \left(\alpha, \frac{\epsilon}{4}, 2 \frac{(1-b^\alpha)^2}{1+b} c\gamma_{\text{new},k}^2 \right) \\ &\quad - 2cF \left(\alpha, \frac{\epsilon}{4}, 4 \frac{b(1-b^{2\alpha})}{1+b} c\gamma_{\text{new},k} \gamma_{\text{new},k-1} \right) \end{aligned} \quad (35)$$

Remark 10.4: d_{-1} corresponds to $a_{t_j+(k-1)\alpha-1}$. When $k=1$, there is no d_{-1} in previous equations, in which case there is no $\frac{b^2(1-b^{2\alpha})}{1-b^2} c\gamma_{\text{new},k-1}^2$ in the upper bound (34).

$$\begin{aligned}
& \sum_{t=0}^{\alpha-1} d_t m'_t \\
&= \sum_{t=0}^{\alpha-1} (b d_{t-1} + \mu_t)(b m_{t-1} + \omega'_t) \\
&= \sum_{t=0}^{\alpha-1} (b^{t+1} d_{-1} + \sum_{i=0}^t b^{t-i} \mu_i)(b^{t+1} m_{-1} + \sum_{i=0}^t b^{t-i} \omega'_i) \\
&= \sum_{t=0}^{\alpha-1} \sum_{i_1=0}^t \sum_{i_2=0}^t b^{2t-i_1-i_2} \mu_{i_1} \omega'_{i_2} + \sum_{t=0}^{\alpha-1} \sum_{i=0}^t b^{2t-i+1} \mu_i m'_{-1} + \sum_{t=0}^{\alpha-1} b^{2(t+1)} d_{-1} m'_{-1} + \\
& \quad \sum_{t=0}^{\alpha-1} \sum_{i=0}^t b^{2t-i+1} d_{-1} \omega'_i \\
&= \sum_{i_1=0}^{\alpha-1} \sum_{t=i_1}^{\alpha-1} \sum_{i_2=0}^t b^{2t-i_1-i_2} \mu_{i_1} \omega'_{i_2} + \sum_{i=0}^{\alpha-1} \sum_{t=i}^{\alpha-1} b^{2t-i+1} \mu_i m'_{-1} + \frac{b^2(1-b^{2\alpha})}{1-b^2} d_{-1} m'_{-1} + \\
& \quad \sum_{i=0}^{\alpha-1} \sum_{t=i}^{\alpha-1} b^{2t-i+1} d_{-1} \omega'_i \\
&= \sum_{i_1=0}^{\alpha-1} \sum_{i_2=0}^{\alpha-1} \sum_{t=\max\{i_1, i_2\}}^{\alpha-1} b^{2t-i_1-i_2} \mu_{i_1} \omega'_{i_2} + \sum_{i=0}^{\alpha-1} \frac{(1-b^{2(\alpha-i)})}{1-b^2} b^{i+1} \mu_i m'_{-1} + \frac{b^2(1-b^{2\alpha})}{1-b^2} d_{-1} m'_{-1} \\
& \quad + \sum_{i=0}^{\alpha-1} \frac{(1-b^{2(\alpha-i)})}{1-b^2} b^{i+1} d_{-1} \omega'_i \\
&= \sum_{i_1=0}^{\alpha-1} \sum_{i_2=0}^{\alpha-1} \frac{(1-b^{2(\alpha-\max\{i_1, i_2\})})}{1-b^2} b^{2\max\{i_1, i_2\}-i_1-i_2} \mu_{i_1} \omega'_{i_2} + \sum_{i=0}^{\alpha-1} \frac{(1-b^{2(\alpha-i)})}{1-b^2} b^{i+1} (\mu_i m'_{-1} + d_{-1} \omega'_i) \\
& \quad + \frac{b^2(1-b^{2\alpha})}{1-b^2} d_{-1} m'_{-1}
\end{aligned} \tag{36}$$

Let

$$\begin{aligned}
Y_{1,i} &= \sum_{i_2=0}^{\alpha-1} \frac{(1-b^{2(\alpha-\max\{i, i_2\})})}{1-b^2} b^{2\max\{i, i_2\}-i-i_2} \mu_i \omega'_{i_2} \\
Y_{2,i} &= \frac{(1-b^{2(\alpha-i)})}{1-b^2} b^{i+1} (\mu_i m'_{-1} + d_{-1} \omega'_i)
\end{aligned}$$

then by Lemma 2.17 and similar procedure to (25),

$$\mathbf{E}_{i-1}(Y_{1,i}|X_{j,k-1}) = 0, \mathbf{E}_{i-1}(Y_{2,i}|X_{j,k-1}) = 0 \tag{37}$$

and

$$\|Y_{1,i_1}\| \leq \sum_{i_2=0}^{\alpha-1} \frac{(1-b^{2(\alpha-\max\{i_1, i_2\})})}{1-b^2} b^{2\max\{i_1, i_2\}-i_1-i_2} \sqrt{cr}(1-b)^2 \gamma_* \gamma_{\text{new},k} \tag{38}$$

As

$$\begin{aligned}
& \sum_{i_2=0}^{\alpha-1} \frac{(1-b^{2(\alpha-\max\{i_1, i_2\})})}{1-b^2} b^{2\max\{i_1, i_2\}-i_1-i_2} \\
&= \sum_{i_2=0}^{i_1} \frac{(1-b^{2(\alpha-i_1)})}{1-b^2} b^{2i_1-i_1-i_2} + \sum_{i_2=i_1+1}^{\alpha-1} \frac{(1-b^{2(\alpha-i_2)})}{1-b^2} b^{2i_2-i_1-i_2} \\
&= \frac{(1-b^{2(\alpha-i_1)}) b^{i_1} (1-b^{-(i_1+1)})}{(1-b^2)(1-b^{-1})} + \frac{b(1-b^{\alpha-1-i_1}) + b^{2(\alpha)-2i_1} (1-b^{-(\alpha-i_1-1)})}{(1-b^2)(1-b)} \\
&= \frac{-b^{i_1+1} + 1 + b^{\alpha-i_1+\alpha+1} + b - b^{\alpha-i_1} - b^{\alpha-i_1+1}}{(1-b^2)(1-b)}
\end{aligned}$$

$$\leq \frac{1+b-2\sqrt{b^{\alpha+1}(1+b-b^{\alpha+1})}}{(1-b^2)(1-b)},$$

we have

$$\|Y_{1,i_1}\|_2 \leq \frac{1+b-2\sqrt{b^{\alpha+1}(1+b-b^{\alpha+1})}}{1+b} \sqrt{cr} \gamma_* \gamma_{\text{new},k}.$$

Also,

$$\begin{aligned} \|Y_{2,i}\|_2 &\leq 2 \frac{(1-b^{2(\alpha-i)})}{1-b^2} b^{+i+1} \sqrt{cr} (1-b) \gamma_* \gamma_{\text{new},k} \\ &\leq 2 \frac{b(1-b^{2\alpha})}{1+b} \sqrt{cr} \gamma_* \gamma_{\text{new},k} \end{aligned} \quad (39)$$

$$\left\| \frac{b^2(1-b^{2\alpha})}{1-b^2} d_{-1} m'_{-1} \right\| \leq \frac{b^2(1-b^{2\alpha})}{1-b^2} \sqrt{cr} \gamma_* \gamma_{\text{new},k} \quad (40)$$

Thus, by Corollary 2.14, we have

$$\mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{i_1 \in \mathcal{I}_{j,k}} Y_{1,i_1} \right\| \leq \epsilon \middle| X_{j,k-1} \right) \geq 1 - (r_j + c) F \left(\alpha, \epsilon, 2 \frac{1+b-2\sqrt{b^{\alpha+1}(1+b-b^{\alpha+1})}}{1+b} \sqrt{cr} \gamma_* \gamma_{\text{new},k} \right), \quad (41)$$

for all $X_{j,k-1} \in \Gamma_{j,k-1}$; and

$$\mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{i_1 \in \mathcal{I}_{j,k}} Y_{2,i_1} \right\| \leq \epsilon \middle| X_{j,k-1} \right) \geq 1 - (r_j + c) F \left(\alpha, \epsilon, 4 \frac{b(1-b^{2\alpha})}{1+b} \sqrt{cr} \gamma_* \gamma_{\text{new},k} \right), \quad (42)$$

for all $X_{j,k-1} \in \Gamma_{j,k-1}$.

Thus, given $X_{j,k-1}$, we have

$$\begin{aligned} &\mathbf{P} \left(\lambda_{\max} \left(\frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} a_{t,\text{new}} a_{t,*}' \right) \leq \frac{b^2(1-b^{2\alpha})}{1-b^2} \sqrt{cr} \gamma_* \gamma_{\text{new},k} + \epsilon \right) \\ &= 1 - \mathbf{P} \left(\lambda_{\max} \left(\frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} a_{t,\text{new}} a_{t,*}' \right) > \frac{b^2(1-b^{2\alpha})}{1-b^2} \sqrt{cr} \gamma_* \gamma_{\text{new},k} + \epsilon \right) \\ &\geq 1 - \mathbf{P} \left(\lambda_{\max} \left(\frac{1}{\alpha} \sum_{i_1 \in \mathcal{I}_{j,k}} Y_{1,i_1} \right) > \frac{\epsilon}{2} \middle| X_{j,k-1} \right) - \mathbf{P} \left(\lambda_{\max} \left(\frac{1}{\alpha} \sum_{i_1 \in \mathcal{I}_{j,k}} Y_{2,i_1} \right) > \frac{\epsilon}{2} \middle| X_{j,k-1} \right) - \\ &\mathbf{P} \left(\lambda_{\max} \left(\frac{1}{\alpha} \sum_{i_1 \in \mathcal{I}_{j,k}} \frac{b^2(1-b^{2\alpha})}{1-b^2} d_{-1} m'_{-1} \right) > \frac{b^2(1-b^{2\alpha})}{1-b^2} \sqrt{cr} \gamma_* \gamma_{\text{new},k} \middle| X_{j,k-1} \right) \\ &\geq 1 - (r_j + c) F \left(\alpha, \frac{\epsilon}{2}, 2 \frac{1+b-2\sqrt{b^{\alpha+1}(1+b-b^{\alpha+1})}}{1+b} \sqrt{cr} \gamma_* \gamma_{\text{new},k} \right) - (r_j + c) F \left(\alpha, \frac{\epsilon}{2}, 4 \frac{b(1-b^{2\alpha})}{1+b} \sqrt{cr} \gamma_* \gamma_{\text{new},k} \right) \quad (43) \end{aligned}$$

$$\begin{aligned} &\mathbf{P} \left(\lambda_{\min} \left(\frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} a_{t,\text{new}} a_{t,*}' \right) \geq -\frac{b^2(1-b^{2\alpha})}{1-b^2} \sqrt{cr} \gamma_* \gamma_{\text{new},k} - \epsilon \right) \\ &= 1 - \mathbf{P} \left(\lambda_{\min} \left(\frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} a_{t,\text{new}} a_{t,*}' \right) < -\frac{b^2(1-b^{2\alpha})}{1-b^2} \sqrt{cr} \gamma_* \gamma_{\text{new},k} - \epsilon \right) \\ &\geq 1 - \mathbf{P} \left(\lambda_{\min} \left(\frac{1}{\alpha} \sum_{i_1 \in \mathcal{I}_{j,k}} Y_{1,i_1} \right) < -\frac{\epsilon}{2} \middle| X_{j,k-1} \right) - \mathbf{P} \left(\lambda_{\min} \left(\frac{1}{\alpha} \sum_{i_1 \in \mathcal{I}_{j,k}} Y_{2,i_1} \right) < -\frac{\epsilon}{2} \middle| X_{j,k-1} \right) - \\ &\mathbf{P} \left(\lambda_{\min} \left(\frac{1}{\alpha} \sum_{i_1 \in \mathcal{I}_{j,k}} \frac{b^2(1-b^{2\alpha})}{1-b^2} d_{-1} m'_{-1} \right) < -\frac{b^2(1-b^{2\alpha})}{1-b^2} \sqrt{cr} \gamma_* \gamma_{\text{new},k} \middle| X_{j,k-1} \right) \geq 1 - \\ &(r_j + c) F \left(\alpha, \frac{\epsilon}{2}, 2 \frac{1+b-2\sqrt{b^{\alpha+1}(1+b-b^{\alpha+1})}}{1+b} \sqrt{cr} \gamma_* \gamma_{\text{new},k} \right) - (r_j + c) F \left(\alpha, \frac{\epsilon}{2}, 4 \frac{b(1-b^{2\alpha})}{1+b} \sqrt{cr} \gamma_* \gamma_{\text{new},k} \right) \quad (44) \end{aligned}$$

$$\begin{aligned}
& \sum_{t=0}^{\alpha-1} m_t m'_t \\
&= \sum_{t=0}^{\alpha-1} (bm_{t-1} + \omega_t)(bm_{t-1} + \omega'_t) \\
&= \sum_{t=0}^{\alpha-1} (b^{t+1}m_{-1} + \sum_{i=0}^t b^{t-i}\omega_i)(b^{t+1}m_{-1} + \sum_{i=0}^t b^{t-i}\omega'_i) \\
&= \sum_{t=0}^{\alpha-1} (b^{2t} \sum_{i=0}^t b^{-2i}\omega_i\omega'_i) + \sum_{t=0}^{\alpha-1} (b^{2t} \sum_{0 \leq i_1, i_2 \leq t, i_1 \neq i_2} b^{-i_1-i_2}\omega_{i_1}\omega'_{i_2}) + \sum_{t=0}^{\alpha-1} \sum_{i=0}^t b^{2t-i+1}m_{-1}\omega'_i \\
&\quad + \sum_{t=0}^{\alpha-1} \sum_{i=0}^t b^{2t-i+1}\omega_i m'_{-1} + \sum_{t=0}^{\alpha-1} b^{2(t+1)}m_{-1}m'_{-1} \\
&= \sum_{i=0}^{\alpha-1} \sum_{t=i}^{\alpha-1} (b^{2t}b^{-2i}\omega_i\omega'_i) + \sum_{t=0}^{\alpha-1} b^{2t} (\sum_{i_1=0}^t \sum_{i_2=0}^{i_1-1} b^{-i_1-i_2}\omega_{i_1}\omega'_{i_2} + \sum_{i_1=0}^t \sum_{i_2=i_1+1}^t b^{-i_1-i_2}\omega_{i_1}\omega'_{i_2}) + \\
&\quad \sum_{i=0}^{\alpha-1} \sum_{t=i}^{\alpha-1} b^{2t-i+1}m_{-1}\omega'_i + \sum_{i=0}^{\alpha-1} \sum_{t=i}^{\alpha-1} b^{2t-i+1}\omega_i m'_{-1} + \frac{b^2(1-b^{2\alpha})}{1-b^2}m_{-1}m'_{-1} \\
&= \sum_{i=0}^{\alpha-1} \frac{(1-b^{2(\alpha-i)})}{1-b^2}\omega_i\omega'_i + \sum_{i_1=0}^{\alpha-1} \sum_{t=i_1}^{\alpha-1} b^{2t} (\sum_{i_2=t_j+(k-1)}^{i_1-1} b^{-i_1-i_2}\omega_{i_1}\omega'_{i_2} + \sum_{i_2=i_1+1}^t b^{-i_1-i_2}\omega_{i_1}\omega'_{i_2}) \\
&\quad \sum_{i=0}^{\alpha-1} \frac{b^{i+1}(1-b^{2(\alpha-i)})}{1-b^2}(\omega_i m'_{-1} + m_{-1}\omega'_i) + \frac{b^2(1-b^{2\alpha})}{1-b^2}m_{-1}m'_{-1} \\
&= \sum_{i=0}^{\alpha-1} \frac{(1-b^{2(\alpha-i)})}{1-b^2}\omega_i\omega'_i + \sum_{i_1=0}^{\alpha-1} \frac{(1-b^{2(\alpha-i_1)})}{1-b^2} \sum_{i_2=0}^{i_1-1} b^{i_1-i_2}\omega_{i_1}\omega'_{i_2} + \sum_{i_1=-1}^{\alpha-1} \sum_{i_2=i_1+1}^{\alpha-1} \sum_{t=i_2}^{\alpha-1} b^{2t}b^{-i_1-i_2}\omega_{i_1}\omega'_{i_2} + \\
&\quad \sum_{i=0}^{\alpha-1} \frac{b^{i+1}(1-b^{2(\alpha-i)})}{1-b^2}(\omega_i m'_{-1} + m_{-1}\omega'_i) + \frac{b^2(1-b^{2\alpha})}{1-b^2}m_{-1}m'_{-1} \\
&= \sum_{i=0}^{\alpha-1} \frac{(1-b^{2(\alpha-i)})}{1-b^2}\omega_i\omega'_i + \sum_{i_1=0}^{\alpha-1} \frac{(1-b^{2(\alpha-i_1)})}{1-b^2} \sum_{i_2=0}^{i_1-1} b^{i_1-i_2}\omega_{i_1}\omega'_{i_2} + \sum_{i_1=0}^{\alpha-1} \sum_{i_2=i_1+1}^{\alpha-1} \frac{(1-b^{2(\alpha-i_2)})}{1-b^2}b^{-i_1+i_2}\omega_{i_1}\omega'_{i_2} \\
&\quad + \sum_{i=0}^{\alpha-1} \frac{b^{i+1}(1-b^{2(\alpha-i)})}{1-b^2}(\omega_i m'_{-1} + m_{-1}\omega'_i) + \frac{b^2(1-b^{2\alpha})}{1-b^2}m_{-1}m'_{-1} \\
&= \sum_{i=0}^{\alpha-1} \frac{(1-b^{2(\alpha-i)})}{1-b^2}\omega_i\omega'_i + \sum_{i_1=0}^{\alpha-1} \frac{(1-b^{2(\alpha-i_1)})}{1-b^2} \sum_{i_2=0}^{i_1-1} b^{i_1-i_2}\omega_{i_1}\omega'_{i_2} + \sum_{i_1=0}^{\alpha-1} \sum_{i_2=\alpha-i_1}^{\alpha-1} \frac{(1-b^{2(\alpha-i_2)})}{1-b^2}b^{i_1+i_2+1-\alpha}\omega_{\alpha-1-i_1}\omega'_{i_2} \\
&\quad + \sum_{i=0}^{\alpha-1} \frac{b^{i+1}(1-b^{2(\alpha-i)})}{1-b^2}(\omega_i m'_{-1} + m_{-1}\omega'_i) + \frac{b^2(1-b^{2\alpha})}{1-b^2}m_{-1}m'_{-1} \tag{45}
\end{aligned}$$

Let

$$\begin{aligned}
Z_{1,i} &= \frac{(1-b^{2(\alpha-i)})}{1-b^2}\omega_i\omega'_i, \\
Z_{2,i} &= \sum_{i_2=0}^{i-1} \frac{(1-b^{2(\alpha-i)})}{1-b^2}b^{i-i_2}\omega_i\omega'_{i_2}, i \geq 1 \\
Z_{3,i} &= \sum_{i_2=\alpha-i}^{\alpha-1} \frac{(1-b^{2(\alpha-i_2)})}{1-b^2}b^{i+i_2+1-\alpha}\omega_{\alpha-1-i}\omega'_{i_2}, \\
Z_{4,i} &= \frac{b^{i+1}(1-b^{2(\alpha-i)})}{1-b^2}\omega_i m'_{-1} \tag{46}
\end{aligned}$$

then,

$$\begin{aligned}
& \lambda_{\max} \left(\mathbf{E}_{i-1} \left(\frac{1}{\alpha} \sum_{i=0}^{\alpha-1} Z_{1,i} \middle| X_{j,k-1} \right) \right) \\
& \leq \frac{1}{\alpha} \sum_{i=0}^{\alpha-1} \frac{(1 - b^{2(t_j + k\alpha - i)})}{1 - b^2} \lambda_{\max}(\mathbf{E}(\omega_i \omega'_i)) \\
& \leq \left(\frac{\alpha(1 - b^2) - b^2 + b^{2\alpha+2}}{\alpha(1 - b^2)} \lambda^+ \right) \\
& \leq \lambda^+ \\
& \lambda_{\min} \left(\mathbf{E}_{i-1} \left(\frac{1}{\alpha} \sum_{i=0}^{\alpha-1} Z_{1,i} \middle| X_{j,k-1} \right) \right) \\
& \geq \frac{1}{\alpha} \sum_{i=0}^{\alpha-1} \frac{(1 - b^{2(t_j + k\alpha - i)})}{1 - b^2} \lambda_{\min}(\mathbf{E}(\omega_i \omega'_i)) \\
& \geq \frac{\alpha(1 - b^2) - b^2 + b^{2\alpha+2}}{\alpha(1 - b^2)} \lambda_{\text{new},k}^-
\end{aligned} \tag{47}$$

and by Lemma 2.17 and similar procedure to (25),

$$\mathbf{E}_{i-1}(Z_{2,i}|X_{j,k-1}) = 0, \mathbf{E}_{i-1}(Z_{3,i}|X_{j,k-1}) = 0, \mathbf{E}_{i-1}(Z_{4,i}|X_{j,k-1}) = 0. \tag{48}$$

$$\begin{aligned}
0 \preceq Z_{1,i_1} & \preceq \frac{(1 - b^{2(\alpha-i)})}{1 - b^2} r(1 - b)^2 \gamma_*^2 I \preceq \frac{(1 - b^{2\alpha})(1 - b)}{1 + b} r \gamma_*^2 I, \\
\|Z_{2,i}\| & \leq \sum_{i_2=0}^{i-1} \frac{(1 - b^{2(\alpha-i)})}{1 - b^2} b^{i-i_2} r(1 - b)^2 \gamma_*^2 \\
& \leq \frac{b(1 - b^{2(\alpha-i)})(1 - b^i)}{1 + b} r \gamma_*^2 \\
& \leq \frac{b(1 - b)(1 - b^{2(\alpha-1)})}{(1 + b)} r \gamma_*^2 \\
\|Z_{3,i}\| & \leq \sum_{i_2=\alpha-i}^{\alpha-1} \frac{(1 - b^{2(\alpha-i_2)})}{1 - b^2} b^{i+i_2+1-\alpha} r(1 - b)^2 \gamma_*^2 \\
& \leq \frac{(1 - b^{i+1})^2}{1 + b} r \gamma_*^2 \\
& \leq \frac{(1 - b^\alpha)^2}{1 + b} r \gamma_*^2 \\
\|Z_{4,i_1}\| & \leq \frac{b^{i+1}(1 - b^{2(\alpha-i)})}{1 - b^2} c(1 - b) \gamma_*^2 \\
& \leq \frac{b(1 - b^{2\alpha})}{1 + b} r \gamma_*^2 \\
0 \preceq \frac{b^2(1 - b^{2\alpha})}{1 - b^2} m_{-1} m'_{-1} & \preceq \frac{b^2(1 - b^{2\alpha})}{1 - b^2} r \gamma_*^2 I
\end{aligned} \tag{49}$$

Given $X_{j,k-1}$, $\{Z_{1,i}\}_{i \in \mathcal{I}_{j,k}}$ is an adapted sequence of Hermitian matrices, thus by Corollary 2.15, for all $X_{j,k-1} \in \Gamma_{j,k-1}$, we have

$$\mathbf{P} \left(\lambda_{\max} \left(\frac{1}{\alpha} \sum_{i=0}^{\alpha-1} Z_{1,i} \right) \leq \lambda^+ - \epsilon \middle| X_{j,k-1} \right) \geq 1 - r_j F(\alpha, \epsilon, \frac{(1 - b^{2\alpha})(1 - b)}{1 + b} r \gamma_*^2) \tag{50}$$

Similarly, given $X_{j,k-1}$, $\{Z_{2,i_1}\}_{i_1 \in \mathcal{I}_{j,k}}$, $\{Z_{3,i_1}\}_{i_1 \in \mathcal{I}_{j,k}}$, $\{Z_{4,i}\}_{i \in \mathcal{I}_{j,k}}$ are adapted sequences of matrices, by Corollary 2.14, for all $X_{j,k-1} \in \Gamma_{j,k-1}$ we have

$$\mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{i=0}^{\alpha-1} Z_{2,i} \right\| \leq \epsilon \middle| X_{j,k-1} \right) \geq 1 - 2r_j F \left(\alpha, \epsilon, 2 \frac{b(1-b)(1-b^{2(\alpha-1)})}{(1+b)} r \gamma_*^2 \right) \quad (51)$$

$$\mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{i=0}^{\alpha-1} Z_{3,i} \right\| \leq \epsilon \middle| X_{j,k-1} \right) \geq 1 - 2r_j F \left(\alpha, \epsilon, 2 \frac{(1-b^\alpha)^2}{1+b} r \gamma_*^2 \right) \quad (52)$$

$$\mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{i=0}^{\alpha-1} (Z_{4,i} + Z'_{4,i}) \right\| \leq \epsilon \middle| X_{j,k-1} \right) \geq 1 - 2r_j F \left(\alpha, \epsilon, 4 \frac{b(1-b^{2\alpha})}{1+b} r \gamma_*^2 \right) \quad (53)$$

Thus, given $X_{j,k-1}$, we have

$$\begin{aligned} & \mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} a_{t,*} a'_{t,*} \right\| \leq \lambda^+ + \frac{b^2(1-b^{2\alpha})}{1-b^2} r \gamma_*^2 + \epsilon \middle| X_{j,k-1} \right) \\ & \geq 1 - \mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} a_{t,*} a'_{t,*} \right\| > \lambda^+ + \frac{b^2(1-b^{2\alpha})}{1-b^2} r \gamma_*^2 + \epsilon \middle| X_{j,k-1} \right) \\ & \geq 1 - \mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{i=0}^{\alpha-1} Z_{1,i} \right\| > \lambda^+ + \frac{\epsilon}{4} \middle| X_{j,k-1} \right) - \mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{i_1=0}^{\alpha-1} Z_{2,i_1} \right\| > \frac{\epsilon}{4} \middle| X_{j,k-1} \right) - \mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{i_1=0}^{\alpha-1} Z_{3,i} \right\| > \frac{\epsilon}{4} \middle| X_{j,k-1} \right) \\ & \quad - \mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{i_1=0}^{\alpha-1} (Z_{4,i} + Z'_{4,i}) \right\| > \frac{\epsilon}{4} \middle| X_{j,k-1} \right) - \mathbf{P} \left(\left\| \frac{b^2(1-b^{2\alpha})}{1-b^2} m_{-1} m'_{-1} \right\| > \frac{b^2(1-b^{2\alpha})}{1-b^2} r \gamma_*^2 \right) \\ & \geq 1 - r_j F \left(\alpha, \frac{\epsilon}{4}, \frac{(1-b^{2\alpha})(1-b)}{1+b} r \gamma_*^2 \right) - 2r_j F \left(\alpha, \frac{\epsilon}{4}, 2 \frac{b(1-b)(1-b^{2(\alpha-1)})}{(1+b)} r \gamma_*^2 \right) \\ & \quad - 2r_j F \left(\alpha, \frac{\epsilon}{4}, 2 \frac{(1-b^\alpha)^2}{1+b} r \gamma_*^2 \right) - 2r_j F \left(\alpha, \frac{\epsilon}{4}, 4 \frac{b(1-b^{2\alpha})}{1+b} r \gamma_*^2 \right) \end{aligned} \quad (54)$$

$$\begin{aligned} & \mathbf{P} \left(\lambda_{\min} \left(\frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} a_{t,*} a'_{t,*} \right) \geq \frac{\alpha(1-b^2) - b^2 + b^{2\alpha+2}}{\alpha(1-b^2)} \lambda_{\text{new},k}^- - \epsilon \middle| X_{j,k-1} \right) \\ & \geq 1 - \mathbf{P} \left(\lambda_{\min} \left(\frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} a_{t,*} a'_{t,*} \right) < \frac{\alpha(1-b^2) - b^2 + b^{2\alpha+2}}{\alpha(1-b^2)} \lambda_{\text{new},k}^- - \epsilon \middle| X_{j,k-1} \right) \\ & \geq 1 - \mathbf{P} \left(\lambda_{\min} \left(\frac{1}{\alpha} \sum_{i=0}^{\alpha-1} Z_{1,i} \right) < \frac{\alpha(1-b^2) - b^2 + b^{2\alpha+2}}{\alpha(1-b^2)} \lambda_{\text{new},k}^- - \frac{\epsilon}{4} \middle| X_{j,k-1} \right) - \mathbf{P} \left(\lambda_{\min} \left(\frac{1}{\alpha} \sum_{i_1=0}^{\alpha-1} Z_{2,i_1} \right) < -\frac{\epsilon}{4} \middle| X_{j,k-1} \right) \\ & \quad - \mathbf{P} \left(\lambda_{\min} \left(\frac{1}{\alpha} \sum_{i_1=0}^{\alpha-1} Z_{3,i} \right) < -\frac{\epsilon}{4} \middle| X_{j,k-1} \right) - \mathbf{P} \left(\lambda_{\min} \left(\frac{1}{\alpha} \sum_{i_1=0}^{\alpha-1} (Z_{4,i} + Z'_{4,i}) \right) < -\frac{\epsilon}{4} \middle| X_{j,k-1} \right) \\ & \quad - \mathbf{P} \left(\lambda_{\min} \left(\frac{b^2(1-b^{2\alpha})}{1-b^2} m_{-1} m'_{-1} \right) < 0 \right) \\ & \geq 1 - r_j F \left(\alpha, \frac{\epsilon}{4}, \frac{(1-b^{2\alpha})(1-b)}{1+b} r \gamma_*^2 \right) - 2r_j F \left(\alpha, \frac{\epsilon}{4}, 2 \frac{b(1-b)(1-b^{2(\alpha-1)})}{(1+b)} r \gamma_*^2 \right) - 2r_j F \left(\alpha, \frac{\epsilon}{4}, 2 \frac{(1-b^\alpha)^2}{1+b} r \gamma_*^2 \right) \\ & \quad - 2r_j F \left(\alpha, \frac{\epsilon}{4}, 4 \frac{b(1-b^{2\alpha})}{1+b} r \gamma_*^2 \right) \end{aligned} \quad (55)$$

B. $\lambda_{\min}(A_k)$

Consider $A_k := \frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} E_{\text{new}}' \Phi_0 L_t L_t' \Phi_0 E_{\text{new}}$. Notice that $E_{\text{new}}' \Phi_0 L_t = R_{\text{new}} a_{t,\text{new}} + E_{\text{new}}' D_* a_{t,*}$. We have

$$A_k \succeq \frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} R_{\text{new}} a_{t,\text{new}} a_{t,\text{new}}' R_{\text{new}}' + \frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} (R_{\text{new}} a_{t,\text{new}} a_{t,*}' D_*' E_{\text{new}}' + E_{\text{new}}' D_* a_{t,*} a_{t,\text{new}}' R_{\text{new}}') \quad (56)$$

By $\lambda_{\min}(R_{\text{new}}R'_{\text{new}}) \geq 1 - (\zeta_*^+)^2$, $\|R_{\text{new}}\| \leq 1$ and similar procedure to get (35), we have

$$\begin{aligned} & \mathbf{P} \left(\lambda_{\min} \left(\frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} R_{\text{new}} a_{t,\text{new}} a'_{t,\text{new}} R'_{\text{new}} \right) \geq (1 - (\zeta_*^+)^2) \left(1 - \frac{b^2 - b^{2\alpha+2}}{\alpha(1-b^2)} \right) \lambda_{\text{new},k}^- - \frac{c\zeta\lambda^-}{24} \middle| X_{j,k-1} \right) \\ & \geq 1 - cF \left(\alpha, \frac{c\zeta\lambda^-}{96}, \frac{(1-b^{2\alpha})(1-b)}{1+b} c\gamma_{\text{new},k}^2 \right) - 2cF \left(\alpha, \frac{c\zeta\lambda^-}{96}, 2 \frac{b(1-b)(1-b^{2(\alpha-1)})}{1+b} c\gamma_{\text{new},k}^2 \right) \\ & \quad - 2cF \left(\alpha, \frac{c\zeta\lambda^-}{96}, 2 \frac{(1-b^\alpha)^2}{1+b} c\gamma_{\text{new},k}^2 \right) - 2cF \left(\alpha, \frac{c\zeta\lambda^-}{96}, 4 \frac{b(1-b^{2\alpha})}{1+b} c\gamma_{\text{new},k} \gamma_{\text{new},k-1} \right) \end{aligned} \quad (57)$$

By $\|R_{\text{new}}\| \leq 1$, $\|E_{\text{new}}\| \leq 1$, $\|D_*\| \leq \zeta_*^+$ and similar procedure to get (44), we have

$$\begin{aligned} & \mathbf{P} \left(\lambda_{\min} \left(\frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} (R_{\text{new}} a_{t,\text{new}} a_{t,*}' D_*' E_{\text{new}}' + E_{\text{new}}' D_* a_{t,*} a_{t,\text{new}}' R_{\text{new}}') \right) \geq -2 \frac{b^2(1-b^{2\alpha})}{1-b^2} \sqrt{c} r \gamma_* \gamma_{\text{new},k} \zeta_*^+ - \frac{c\zeta\lambda^-}{24} \right) \geq \\ & 1 - 2cF \left(\alpha, \frac{c\zeta\lambda^-}{48}, 4 \frac{1+b-2\sqrt{b^{\alpha+1}(1+b-b^{\alpha+1})}}{1+b} \sqrt{c} r \gamma_* \gamma_{\text{new},k} \zeta_*^+ \right) - 2cF \left(\alpha, \frac{c\zeta\lambda^-}{48}, 8 \frac{b(1-b^{2\alpha})}{1+b} \sqrt{c} r \gamma_* \gamma_{\text{new},k} \zeta_*^+ \right) \end{aligned} \quad (58)$$

Combining the previous two inequalities, we have,

$$\mathbf{P} \left(\lambda_{\min}(A_k) \geq b_{A_k} - \frac{c\zeta\lambda^-}{12} \right) \geq 1 - p_a(\alpha, \zeta) \quad (59)$$

where $b_{A_k} = (1 - (\zeta_*^+)^2) \left(1 - \frac{b^2 - b^{2\alpha+2}}{\alpha(1-b^2)} \right) \lambda_{\text{new},k}^- - 2 \frac{b^2(1-b^{2\alpha})}{1-b^2} \sqrt{c} r \gamma_* \gamma_{\text{new},k} \zeta_*^+$ and

$$\begin{aligned} p_a(\alpha, \zeta) = & cF \left(\alpha, \frac{c\zeta\lambda^-}{96}, \frac{(1-b^{2\alpha})(1-b)}{1+b} c\gamma_{\text{new},k}^2 \right) + 2cF \left(\alpha, \frac{c\zeta\lambda^-}{96}, 2 \frac{b(1-b)(1-b^{2(\alpha-1)})}{1+b} c\gamma_{\text{new},k}^2 \right) \\ & + 2cF \left(\alpha, \frac{c\zeta\lambda^-}{96}, 2 \frac{(1-b^\alpha)^2}{1+b} c\gamma_{\text{new},k}^2 \right) + 2cF \left(\alpha, \frac{c\zeta\lambda^-}{96}, 4 \frac{b(1-b^{2\alpha})}{1+b} c\gamma_{\text{new},k} \gamma_{\text{new},k-1} \right) + 2cF \left(\alpha, \frac{c\zeta\lambda^-}{48}, \right. \\ & \left. 4 \frac{1+b-2\sqrt{b^{\alpha+1}(1+b-b^{\alpha+1})}}{1+b} \sqrt{c} r \gamma_* \gamma_{\text{new},k} \zeta_*^+ \right) + 2cF \left(\alpha, \frac{c\zeta\lambda^-}{48}, 8 \frac{b(1-b^{2\alpha})}{1+b} \sqrt{c} r \gamma_* \gamma_{\text{new},k} \zeta_*^+ \right) \end{aligned} \quad (60)$$

C. $\lambda_{\max}(A_{k,\perp})$

$$\begin{aligned} A_{k,\perp} &:= \frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} E_{\text{new},\perp}' \Phi_0 L_t L_t' \Phi_0 E_{\text{new},\perp} \\ &= \frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} E_{\text{new},\perp}' D_* a_{t,*} a'_{t,*} D_*' E_{\text{new},\perp} \end{aligned}$$

By $\|E_{\text{new},\perp}\| \leq 1$, $\|D_*\| \leq \zeta_*^+$ and similar procedure to get (54), we have

$$\mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} E_{\text{new},\perp}' D_* a_{t,*} a'_{t,*} D_*' E_{\text{new},\perp} \right\| \leq b_{A_{k,\perp}} + \frac{c\zeta\lambda^-}{24} \middle| X_{j,k-1} \right) \geq 1 - p_b(\alpha, \zeta) \quad (61)$$

where $b_{A_{k,\perp}} = \lambda^+(\zeta_*^+)^2 + \frac{b^2(1-b^{2\alpha})}{1-b^2} r \gamma_*^2 (\zeta_*^+)^2$ and

$$\begin{aligned} p_b(\alpha, \zeta) = & (n-c)F \left(\alpha, \frac{c\zeta\lambda^-}{96}, \frac{(1-b^{2\alpha})(1-b)}{1+b} r \gamma_*^2 (\zeta_*^+)^2 \right) + 2(n-c)F \left(\alpha, \frac{c\zeta\lambda^-}{96}, 2 \frac{b(1-b)(1-b^{2(\alpha-1)})}{(1+b)} r \gamma_*^2 (\zeta_*^+)^2 \right) \\ & + 2(n-c)F \left(\alpha, \frac{c\zeta\lambda^-}{96}, 2 \frac{(1-b^\alpha)^2}{1+b} r \gamma_*^2 (\zeta_*^+)^2 \right) + 2(n-c)F \left(\alpha, \frac{c\zeta\lambda^-}{96}, 4 \frac{b(1-b^{2\alpha})}{1+b} r \gamma_*^2 (\zeta_*^+)^2 \right) \end{aligned} \quad (62)$$

D. $\|\mathcal{H}_k\|_2$

In this proof, we frequently refer to items from Sec. 10.3 and the bounds (22).

For the second claim, using the expression for \mathcal{H}_k given in Definition 6.5, it is easy to see that

$$\|\mathcal{H}_k\|_2 \leq \max\{\|H_k\|_2, \|H_{k,\perp}\|_2\} + \|B_k\|_2 \leq \left\| \frac{1}{\alpha} \sum_t e_t e_t' \right\|_2 + \max(\|T^2\|_2, \|T^4\|_2) + \|B_k\|_2 \quad (63)$$

where $T2 := \frac{1}{\alpha} \sum_t E_{\text{new}}' \Phi_0 (L_t e_t' + e_t L_t') \Phi_0 E_{\text{new}}$ and $T4 := \frac{1}{\alpha} \sum_t E_{\text{new},\perp}' \Phi_0 (L_t e_t' + e_t L_t') \Phi_0 E_{\text{new},\perp}$. The second inequality follows by using the facts that (i) $H_k = T1 - T2$ where $T1 := \frac{1}{\alpha} \sum_t E_{\text{new}}' \Phi_0 e_t e_t' \Phi_0 E_{\text{new}}$, (ii) $H_{k,\perp} = T3 - T4$ where $T3 := \frac{1}{\alpha} \sum_t E_{\text{new},\perp}' \Phi_0 e_t e_t' \Phi_0 E_{\text{new},\perp}$, and (iii) $\max(\|T1\|_2, \|T3\|_2) \leq \|\frac{1}{\alpha} \sum_t e_t e_t'\|_2$. Next, we obtain high probability bounds on each of the terms on the RHS of (63) using the Azuma corollaries.

I) $\|\frac{1}{\alpha} \sum_t e_t e_t'\|_2$: Consider $\|\frac{1}{\alpha} \sum_t e_t e_t'\|_2$. Let $Z_t = e_t e_t'$. Then

$$\begin{aligned} Z_t &= I_{T_t} [(\Phi_{k-1})_{T_t}' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' [(\Phi_{k-1} P_*) a_{t,*} + D_{\text{new},k-1} a_{t,\text{new}}] [(\Phi_{k-1} P_*) a_{t,*} + D_{\text{new},k-1} a_{t,\text{new}}]' I_{T_t} [(\Phi_{k-1})_{T_t}' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' \\ &= I_{T_t} [(\Phi_{k-1})_{T_t}' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' [(\Phi_{k-1} P_*) a_{t,*} a_{t,*}' P_*' \Phi_{k-1} + D_{\text{new},k-1} a_{t,\text{new}} a_{t,\text{new}}' D_{\text{new},k-1}' + (\Phi_{k-1} P_*) a_{t,*} a_{t,\text{new}}' D_{\text{new},k-1}' \\ &\quad + D_{\text{new},k-1} a_{t,\text{new}} a_{t,*}' P_*' \Phi_{k-1}] I_{T_t} [(\Phi_{k-1})_{T_t}' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' \end{aligned} \quad (64)$$

(a) By (22) and similar procedure to get (54), we have

$$\|I_{T_t} [(\Phi_{k-1})_{T_t}' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' (\Phi_{k-1} P_*)\| \leq \phi^+ \zeta_*^+$$

and

$$\begin{aligned} \mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} I_{T_t} [(\Phi_{k-1})_{T_t}' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' [(\Phi_{k-1} P_*) a_{t,*} a_{t,*}' P_*' \Phi_{k-1} I_{T_t} [(\Phi_{k-1})_{T_t}' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' \right\| \right. \\ \left. \leq \lambda^+ (\phi^+ \zeta_*^+)^2 + \frac{b^2(1-b^{2\alpha})}{1-b^2} r \gamma_*^2 (\phi^+ \zeta_*^+)^2 + \frac{c\zeta \lambda^-}{72} |X_{j,k-1}| \right) \\ \geq 1 - p_{c_1}(\alpha, \zeta) \end{aligned} \quad (65)$$

where

$$\begin{aligned} p_{c_1}(\alpha, \zeta) &= nF \left(\alpha, \frac{c\zeta \lambda^-}{288}, \frac{(1-b^{2\alpha})(1-b)}{1+b} r \gamma_*^2 (\phi^+ \zeta_*^+)^2 \right) + 2nF \left(\alpha, \frac{c\zeta \lambda^-}{288}, 2 \frac{b(1-b)(1-b^{2(\alpha-1)})}{(1+b)} r \gamma_*^2 (\phi^+ \zeta_*^+)^2 \right) \\ &\quad + 2nF \left(\alpha, \frac{c\zeta \lambda^-}{288}, 2 \frac{(1-b^\alpha)^2}{1+b} r \gamma_*^2 (\phi^+ \zeta_*^+)^2 \right) + 2nF \left(\alpha, \frac{c\zeta \lambda^-}{288}, 4 \frac{b(1-b^{2\alpha})}{1+b} r \gamma_*^2 (\phi^+ \zeta_*^+)^2 \right) \end{aligned} \quad (66)$$

(b) By (22) and similar procedure to get (34), we have

$$\|I_{T_t} [(\Phi_{k-1})_{T_t}' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' D_{\text{new},k-1}\| \leq \phi^+ \kappa_s^+ \zeta_{k-1}^+$$

and

$$\begin{aligned} \mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} I_{T_t} [(\Phi_{k-1})_{T_t}' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' D_{\text{new},k-1} a_{t,\text{new}} a_{t,\text{new}}' D_{\text{new},k-1}' I_{T_t} [(\Phi_{k-1})_{T_t}' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' \right\| \right. \\ \left. \leq \lambda_{\text{new},k}^+ (\phi^+ \kappa_s^+ \zeta_{k-1}^+)^2 + \frac{b^2(1-b^{2\alpha})}{1-b^2} c \gamma_{\text{new},k-1}^2 (\phi^+ \kappa_s^+ \zeta_{k-1}^+)^2 + \frac{c\zeta \lambda^-}{72} |X_{j,k-1}| \right) \\ \geq 1 - p_{c_2}(\alpha, \zeta) \end{aligned} \quad (67)$$

where

$$\begin{aligned} p_{c_2}(\alpha, \zeta) &= nF \left(\alpha, \frac{c\zeta \lambda^-}{288}, \frac{(1-b^{2\alpha})(1-b)}{1+b} c \gamma_{\text{new},k}^2 (\phi^+ \kappa_s^+ \zeta_{k-1}^+)^2 \right) + 2nF \left(\alpha, \frac{c\zeta \lambda^-}{288}, 2 \frac{b(1-b)(1-b^{2(\alpha-1)})}{1+b} c \gamma_{\text{new},k}^2 (\phi^+ \kappa_s^+ \zeta_{k-1}^+)^2 \right) \\ &\quad + 2nF \left(\alpha, \frac{c\zeta \lambda^-}{288}, 2 \frac{(1-b^\alpha)^2}{1+b} c \gamma_{\text{new},k}^2 (\phi^+ \kappa_s^+ \zeta_{k-1}^+)^2 \right) + 2nF \left(\alpha, \frac{c\zeta \lambda^-}{288}, 4 \frac{b(1-b^{2\alpha})}{1+b} c \gamma_{\text{new},k} \gamma_{\text{new},k-1} (\phi^+ \kappa_s^+ \zeta_{k-1}^+)^2 \right) \end{aligned}$$

(c) By (22) and similar procedure to get (43), we have

$$\|I_{T_t} [(\Phi_{k-1})_{T_t}' (\Phi_{k-1})_{T_t}]^{-1}\| \leq \phi^+, \|I_{T_t}' D_{\text{new},k-1}\| \leq \kappa_s^+ \zeta_{k-1}^+, \|\Phi_{k-1} P_*\| \leq \zeta_*^+$$

and

$$\mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} I_{T_t} [(\Phi_{k-1})_{T_t}' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' [(\Phi_{k-1} P_*) a_{t,*} a_{t,\text{new}}' D_{\text{new},k-1}' + D_{\text{new},k-1} a_{t,\text{new}} a_{t,*}' P_*' \Phi_{k-1}] \right\| \right)$$

$$\begin{aligned}
& \left\| I_{T_t} [(\Phi_{k-1})_{T_t}]' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' \right\| \leq 2 \frac{b^2 (1 - b^{2\alpha})}{1 - b^2} \sqrt{c r \gamma_* \gamma_{\text{new},k}} (\phi^+)^2 \kappa_s^+ \zeta_{k-1}^+ \zeta_*^+ + \frac{c \zeta \lambda^-}{72} \left| X_{j,k-1} \right| \\
& \geq 1 - p_{c_3}(\alpha, \zeta)
\end{aligned} \tag{68}$$

where

$$\begin{aligned}
p_{c_3}(\alpha, \zeta) = & 2nF\left(\alpha, \frac{c\zeta\lambda^-}{144}, 4 \frac{1+b-2\sqrt{b^{\alpha+1}(1+b-b^{\alpha+1})}}{1+b} \sqrt{c r \gamma_* \gamma_{\text{new},k}} (\phi^+)^2 \kappa_s^+ \zeta_{k-1}^+ \zeta_*^+ \right) + \\
& 2nF\left(\alpha, \frac{c\zeta\lambda^-}{144}, 8 \frac{b(1-b^{2\alpha})}{1+b} \sqrt{c r \gamma_* \gamma_{\text{new},k}} (\phi^+)^2 \kappa_s^+ \zeta_{k-1}^+ \zeta_*^+ \right)
\end{aligned} \tag{69}$$

Combining (65), (67) and (68), we have

$$\begin{aligned}
\mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} e_t e_t' \right\| \leq \lambda^+ (\phi^+ \zeta_*^+)^2 + \frac{b^2(1-b^{2\alpha})}{1-b^2} r \gamma_*^2 (\phi^+ \zeta_*^+)^2 + \lambda_{\text{new},k}^+ (\phi^+ \kappa_s^+ \zeta_{k-1}^+)^2 + \frac{b^2(1-b^{2\alpha})}{1-b^2} c \gamma_{\text{new},k-1}^2 (\phi^+ \kappa_s^+ \zeta_{k-1}^+)^2 + \right. \\
\left. 2 \frac{b^2(1-b^{2\alpha})}{1-b^2} \sqrt{c r \gamma_* \gamma_{\text{new},k}} (\phi^+)^2 \kappa_s^+ \zeta_{k-1}^+ \zeta_*^+ + \frac{c \zeta \lambda^-}{24} \left| X_{j,k-1} \right| \right) \geq 1 - p_c(\alpha, \zeta)
\end{aligned} \tag{70}$$

where $p_c(\alpha, \zeta) = p_{c_1}(\alpha, \zeta) + p_{c_2}(\alpha, \zeta) + p_{c_3}(\alpha, \zeta)$.

2) $\|T_2\|$: Consider T_2 . Let

$$\begin{aligned}
Z_t &:= E_{\text{new}}' \Phi_0 L_t e_t' \Phi_0 E_{\text{new}} \\
&= E_{\text{new}}' \Phi_0 (P_* a_{t,*} + P_{\text{new}} a_{t,\text{new}}) [(\Phi_{k-1} P_*) a_{t,*} + D_{\text{new},k-1} a_{t,\text{new}}]' I_{T_t} [(\Phi_{k-1})_{T_t}]' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' \Phi_0 E_{\text{new}} \\
&= E_{\text{new}}' \Phi_0 (P_* a_{t,*} a_{t,*}' P_*' \Phi_{k-1} + P_{\text{new}} a_{t,\text{new}} a_{t,*}' P_*' \Phi_{k-1} + P_* a_{t,*} a_{t,\text{new}}' D_{\text{new},k-1}' + P_{\text{new}} a_{t,\text{new}} a_{t,\text{new}}' D_{\text{new},k-1}') I_{T_t} \\
&\quad [(\Phi_{k-1})_{T_t}]' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' \Phi_0 E_{\text{new}}
\end{aligned} \tag{71}$$

which is of size $c \times c$. Then $T_2 = \frac{1}{\alpha} \sum_t (Z_t + Z_t')$.

(a) By (22) and similar procedure to get (54), we have

$$\|E_{\text{new}}' \Phi_0 P_*\| \leq \zeta_*^+, \|P_*' \Phi_{k-1} I_{T_t} [(\Phi_{k-1})_{T_t}]' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' \Phi_0 E_{\text{new}}\| \leq \zeta_*^+ \phi^+ \frac{\kappa_s^+}{\sqrt{1 - (\zeta_*^+)^2}}$$

and

$$\begin{aligned}
\mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} E_{\text{new}}' \Phi_0 P_* a_{t,*} a_{t,*}' P_*' \Phi_{k-1} I_{T_t} [(\Phi_{k-1})_{T_t}]' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' \Phi_0 E_{\text{new}} \right\| \right. \\
\left. \leq \lambda^+ \phi^+ (\zeta_*^+)^2 \frac{\kappa_s^+}{\sqrt{1 - (\zeta_*^+)^2}} + \frac{b^2(1-b^{2\alpha})}{1-b^2} r \gamma_*^2 \phi^+ (\zeta_*^+)^2 \frac{\kappa_s^+}{\sqrt{1 - (\zeta_*^+)^2}} + \frac{c \zeta \lambda^-}{144} \left| X_{j,k-1} \right| \right) \\
\geq 1 - p_{d_1}(\alpha, \zeta)
\end{aligned} \tag{72}$$

where

$$\begin{aligned}
p_{d_1}(\alpha, \zeta) = & cF \left(\alpha, \frac{c\zeta\lambda^-}{576}, \frac{(1-b^{2\alpha})(1-b)}{1+b} r \gamma_*^2 \phi^+ (\zeta_*^+)^2 \frac{\kappa_s^+}{\sqrt{1 - (\zeta_*^+)^2}} \right) + 2cF \left(\alpha, \frac{c\zeta\lambda^-}{576}, 2 \frac{b(1-b)(1-b^{2(\alpha-1)})}{(1+b)} r \gamma_*^2 \phi^+ (\zeta_*^+)^2 \frac{\kappa_s^+}{\sqrt{1 - (\zeta_*^+)^2}} \right) \\
& + 2cF \left(\alpha, \frac{c\zeta\lambda^-}{576}, 2 \frac{(1-b^\alpha)^2}{1+b} r \gamma_*^2 \phi^+ (\zeta_*^+)^2 \frac{\kappa_s^+}{\sqrt{1 - (\zeta_*^+)^2}} \right) + 2cF \left(\alpha, \frac{c\zeta\lambda^-}{576}, 4 \frac{b(1-b^{2\alpha})}{1+b} r \gamma_*^2 \phi^+ (\zeta_*^+)^2 \frac{\kappa_s^+}{\sqrt{1 - (\zeta_*^+)^2}} \right)
\end{aligned} \tag{73}$$

(b) By (22) and similar procedure to get (43), we have

$$\|E_{\text{new}}' \Phi_0 P_*\| \leq \zeta_*^+ \leq 1, \|E_{\text{new}}' \Phi_0 P_{\text{new}}\| \leq 1, \|P_*' \Phi_{k-1} I_{T_t} [(\Phi_{k-1})_{T_t}]' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' \Phi_0 E_{\text{new}}\| \leq \zeta_*^+ \phi^+ \frac{\kappa_s^+}{\sqrt{1 - (\zeta_*^+)^2}}$$

and

$$\begin{aligned} \mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} E_{\text{new}}' \Phi_0 (P_{\text{new}} a_{t,\text{new}} a'_{t,*} P_*' \Phi_{k-1} + P_* a_{t,*} a'_{t,\text{new}} D'_{\text{new},k-1}) I_{T_t} [(\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' \Phi_0 E_{\text{new}} \right\| \right. \\ \left. \leq 2 \frac{b^2 (1 - b^{2\alpha})}{1 - b^2} \sqrt{c r \gamma_* \gamma_{\text{new},k}} \phi^+ \zeta_*^+ \frac{\kappa_s^+}{\sqrt{1 - (\zeta_*^+)^2}} + \frac{c \zeta \lambda^-}{144} \right) \geq 1 - p_{d_3}(\alpha, \zeta) \end{aligned} \quad (74)$$

where

$$\begin{aligned} p_{d_3}(\alpha, \zeta) = 2cF(\alpha, \frac{c\zeta\lambda^-}{288}, 4 \frac{1+b-2\sqrt{b^{\alpha+1}(1+b-b^{\alpha+1})}}{1+b}) \sqrt{c r \gamma_* \gamma_{\text{new},k}} \phi^+ \zeta_*^+ \frac{\kappa_s^+}{\sqrt{1 - (\zeta_*^+)^2}} \\ + 2cF \left(\alpha, \frac{c\zeta\lambda^-}{288}, 8 \frac{b(1-b^{2\alpha})}{1+b} \sqrt{c r \gamma_* \gamma_{\text{new},k}} \phi^+ \zeta_*^+ \frac{\kappa_s^+}{\sqrt{1 - (\zeta_*^+)^2}} \right) \end{aligned} \quad (75)$$

(c) By (22) and similar procedure to get (34), we have

$$\|E_{\text{new}}' \Phi_0 P_{\text{new}}\| \leq 1, \|D'_{\text{new},k-1} I_{T_t} [(\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' \Phi_0 E_{\text{new}}\| \leq \zeta_{k-1}^+ \phi^+ \frac{(\kappa_s^+)^2}{\sqrt{1 - (\zeta_*^+)^2}}$$

and

$$\begin{aligned} \mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{t \in \mathcal{I}_{j,k}} I_{T_t} [(\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' D_{\text{new},k-1} a_{t,\text{new}} a'_{t,\text{new}} D'_{\text{new},k-1} I_{T_t} [(\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' \right\| \right. \\ \left. \leq \lambda_{\text{new},k}^+ \zeta_{k-1}^+ \phi^+ \frac{(\kappa_s^+)^2}{\sqrt{1 - (\zeta_*^+)^2}} + \frac{b^2 (1 - b^{2\alpha})}{1 - b^2} c \gamma_{\text{new},k-1}^2 \zeta_{k-1}^+ \phi^+ \frac{(\kappa_s^+)^2}{\sqrt{1 - (\zeta_*^+)^2}} + \frac{c \zeta \lambda^-}{144} |X_{j,k-1}| \right) \\ \geq 1 - p_{d_3}(\alpha, \zeta) \end{aligned} \quad (76)$$

where

$$\begin{aligned} p_{d_3}(\alpha, \zeta) = cF \left(\alpha, \frac{c\zeta\lambda^-}{576}, \frac{(1-b^{2\alpha})(1-b)}{1+b} c \gamma_{\text{new},k}^2 (\phi^+ \kappa_s^+ \zeta_{k-1}^+)^2 \frac{(\kappa_s^+)^2}{\sqrt{1 - (\zeta_*^+)^2}} \right) \\ + 2cF \left(\alpha, \frac{c\zeta\lambda^-}{576}, 2 \frac{b(1-b)(1-b^{2(\alpha-1)})}{1+b} c \gamma_{\text{new},k}^2 (\phi^+ \kappa_s^+ \zeta_{k-1}^+)^2 \frac{(\kappa_s^+)^2}{\sqrt{1 - (\zeta_*^+)^2}} \right) \\ + 2cF \left(\alpha, \frac{c\zeta\lambda^-}{576}, 2 \frac{(1-b^\alpha)^2}{1+b} c \gamma_{\text{new},k}^2 (\phi^+ \kappa_s^+ \zeta_{k-1}^+)^2 \frac{(\kappa_s^+)^2}{\sqrt{1 - (\zeta_*^+)^2}} \right) \\ + 2cF \left(\alpha, \frac{c\zeta\lambda^-}{576}, 4 \frac{b(1-b^{2\alpha})}{1+b} c \gamma_{\text{new},k} \gamma_{\text{new},k-1} (\phi^+ \kappa_s^+ \zeta_{k-1}^+)^2 \frac{(\kappa_s^+)^2}{\sqrt{1 - (\zeta_*^+)^2}} \right) \end{aligned} \quad (77)$$

Thus, combining (72), (74) and (76), we have

$$\begin{aligned} \mathbf{P} \left(\|T2\| = \left\| \frac{1}{\alpha} \sum_t (Z_t + Z_t') \right\| \leq 2\phi^+ \kappa_s^+ \frac{(\zeta_*^+)^2}{\sqrt{1 - (\zeta_*^+)^2}} \lambda^+ + 2\phi^+ \zeta_{k-1}^+ \frac{(\kappa_s^+)^2}{\sqrt{1 - (\zeta_*^+)^2}} \lambda_{\text{new},k}^+ + \right. \\ \left. 2 \frac{b^2 (1 - b^{2\alpha})}{1 - b^2} r \gamma_*^2 \phi^+ \kappa_s^+ \frac{(\zeta_*^+)^2}{\sqrt{1 - (\zeta_*^+)^2}} + 2 \frac{b^2 (1 - b^{2\alpha})}{1 - b^2} c \gamma_{\text{new},k-1}^2 \phi^+ \zeta_{k-1}^+ \frac{(\kappa_s^+)^2}{\sqrt{1 - (\zeta_*^+)^2}} + \right. \\ \left. 4 \frac{b^2 (1 - b^{2\alpha})}{1 - b^2} \sqrt{c r \gamma_* \gamma_{\text{new},k}} \zeta_*^+ \phi^+ \frac{\kappa_s^+}{\sqrt{1 - (\zeta_*^+)^2}} + \frac{c \zeta \lambda^-}{24} |X_{j,k-1}| \right) \\ \geq 1 - p_d(\alpha, \zeta) \end{aligned} \quad (78)$$

where $p_d(\alpha, \zeta) = 2p_{d_1}(\alpha, \zeta) + 2p_{d_2}(\alpha, \zeta) + 2p_{d_3}(\alpha, \zeta)$.

3) $\|T_4\|$: Consider T_4 . Let

$$Z_t := E_{\text{new},\perp} {}' \Phi_0 L_t e_t {}' \Phi_0 E_{\text{new},\perp} \quad (79)$$

which is of size $(n-c) \times (n-c)$. Then $T_4 = \frac{1}{\alpha} \sum_t (Z_t + Z_t')$. $E_{\text{new},\perp} {}' \Phi_0 L_t = E_{\text{new},\perp} {}' \Phi_0 P_* a_{t,*} + E_{\text{new},\perp} {}' E_{\text{new}} R_{\text{new}} a_{t,\text{new}} = E_{\text{new},\perp} {}' D_* a_{t,*}$. Thus,

$$\begin{aligned} Z_t &= E_{\text{new},\perp} {}' D_* a_{t,*} [(\Phi_{k-1} P_*) a_{t,*} + D_{\text{new},k-1} a_{t,\text{new}}] {}' I_{T_t} [(\Phi_{k-1})_{T_t}] {}' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t} {}' \Phi_0 E_{\text{new},\perp} \\ &= E_{\text{new},\perp} {}' D_* a_{t,*} (a'_{t,*} P_*' \Phi_{k-1} + a'_{t,\text{new}} D'_{\text{new},k-1}) I_{T_t} [(\Phi_{k-1})_{T_t}] {}' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t} {}' \Phi_0 E_{\text{new},\perp} \end{aligned}$$

(a) Using (22) and similar procedure to get (54), we have

$$\begin{aligned} \mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{t \in t_{j,k}} E_{\text{new},\perp} {}' D_* a_{t,*} a'_{t,*} P_*' \Phi_{k-1} I_{T_t} [(\Phi_{k-1})_{T_t}] {}' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t} {}' \Phi_0 E_{\text{new},\perp} \right\|_2 \leq \phi^+ (\zeta_*^+)^2 \lambda^+ + \right. \\ \left. \frac{b^2 (1-b^{2\alpha})}{1-b^2} r \gamma_*^2 \phi^+ (\zeta_*^+)^2 + \frac{c \zeta \lambda^-}{96} \Big| X_{j,k-1} \right) \geq 1 - p_{e_1}(\alpha, \zeta) \end{aligned} \quad (80)$$

where

$$\begin{aligned} p_{e_1}(\alpha, \zeta) &= 2(n-c)F \left(\alpha, \frac{c \zeta \lambda^-}{384}, \frac{(1-b^{2\alpha})(1-b)}{1+b} r \gamma_*^2 \phi^+ (\zeta_*^+)^2 \right) + 2(n-c)F \left(\alpha, \frac{c \zeta \lambda^-}{384}, 2 \frac{b(1-b)(1-b^{2(\alpha-1)})}{1+b} r \gamma_*^2 \phi^+ \right. \\ &\quad \left. (\zeta_*^+)^2 \right) + 2(n-c)F \left(\alpha, \frac{c \zeta \lambda^-}{384}, 2 \frac{(1-b^{\alpha})^2}{1+b} r \gamma_*^2 \phi^+ (\zeta_*^+)^2 \right) + 2(n-c)F \left(\alpha, \frac{c \zeta \lambda^-}{384}, 2 \frac{b(1-b^{2\alpha})}{1+b} r \gamma_*^2 \phi^+ (\zeta_*^+)^2 \right) \end{aligned}$$

(b) Using (22) and similar procedure to get (43), we have, conditioned on $X_{j,k-1}$,

$$\begin{aligned} \mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{t \in t_{j,k}} E_{\text{new},\perp} {}' D_* a_{t,*} a'_{t,\text{new}} D'_{\text{new},k-1} I_{T_t} [(\Phi_{k-1})_{T_t}] {}' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t} {}' \Phi_0 E_{\text{new},\perp} \right\|_2 \leq \right. \\ \left. \frac{b^2 (1-b^{2\alpha})}{1-b^2} \sqrt{c r \gamma_* \gamma_{\text{new},k}} \zeta_*^+ \zeta_{k-1}^+ \phi^+ \kappa_s^+ + \frac{c \zeta \lambda^-}{96} \Big| X_{j,k-1} \right) \geq 1 - p_{e_2}(\alpha, \zeta) \end{aligned} \quad (81)$$

where

$$\begin{aligned} p_{e_2}(\alpha, \zeta) &= 2(n-c)F \left(\alpha, \frac{c \zeta \lambda^-}{192}, 2 \frac{1+b-2\sqrt{b^{\alpha+1}(1+b-b^{\alpha+1})}}{1+b} \sqrt{c r \gamma_* \gamma_{\text{new},k}} \zeta_*^+ \zeta_{k-1}^+ \phi^+ \kappa_s^+ \right) \\ &\quad + 2(n-c)F \left(\alpha, \frac{c \zeta \lambda^-}{192}, 4 \frac{b(1-b^{2\alpha})}{1+b} \sqrt{c r \gamma_* \gamma_{\text{new},k}} \zeta_*^+ \zeta_{k-1}^+ \phi^+ \kappa_s^+ \right) \end{aligned}$$

Thus, combining last two inequalities, we have

$$\begin{aligned} \mathbf{P} \left(\|T_4\|_2 = \left\| \frac{1}{\alpha} \sum_{t \in t_{j,k}} Z_t + Z_t' \right\|_2 \leq 2\phi^+ (\zeta_*^+)^2 \lambda^+ + 2 \frac{b^2 (1-b^{2\alpha})}{1-b^2} r \gamma_*^2 \phi^+ (\zeta_*^+)^2 + 2 \frac{b^2 (1-b^{2\alpha})}{1-b^2} \sqrt{c r \gamma_* \gamma_{\text{new},k}} \zeta_*^+ \zeta_{k-1}^+ \phi^+ \kappa_s^+ + \right. \\ \left. \frac{c \zeta \lambda^-}{24} \Big| X_{j,k-1} \right) \geq 1 - p_e(\alpha, \zeta) \end{aligned} \quad (82)$$

where

$$p_e(\alpha, \zeta) = 2p_{e_1}(\alpha, \zeta) + 2p_{e_2}(\alpha, \zeta) \quad (83)$$

By condition in Theorem 18, $\zeta_*^+ = r_0 \zeta + (j-1)c\zeta \leq r\zeta$, $\kappa_s^+ = 0.15$, $\zeta_*^+ r f < 1.5 \times 10^{-4}$, we have

$$\begin{aligned} r \gamma_*^2 (\zeta_*^+)^2 \leq (\zeta_*^+)^2 \lambda^+ \eta / c = (\zeta_*^+)^2 f \lambda_{\text{new},k}^- \eta / c = \zeta_*^+ f r \zeta \lambda_{\text{new},k}^- \eta / c \leq 1.5 \times 10^{-4} \zeta \lambda_{\text{new},k}^- \eta \\ < 0.15 c \zeta \frac{\kappa_s^+}{\sqrt{1-(\zeta_*^+)^2}} \eta \lambda_{\text{new},k}^- \leq \max\{0.15 c \zeta, \zeta_{k-1}^+\} \frac{\kappa_s^+}{\sqrt{1-(\zeta_*^+)^2}} \eta \lambda_{\text{new},k}^- \end{aligned} \quad (84)$$

Thus,

$$\mathbf{P} \left(\max\{\|T_2\|_2, \|T_4\|_2\} \leq 2\phi^+ \kappa_s^+ \frac{(\zeta_*^+)^2}{\sqrt{1-(\zeta_*^+)^2}} \left(\lambda^+ + \frac{b^2(1-b^{2\alpha})}{1-b^2} r \gamma_*^2 \right) + 2\phi^+ \max\{0.15 c \zeta, \zeta_{k-1}^+\} \frac{(\kappa_s^+)^2}{\sqrt{1-(\zeta_*^+)^2}} \left(\lambda_{\text{new},k}^+ \right. \right.$$

$$+ \frac{b^2(1-b^{2\alpha})}{1-b^2} \eta \lambda_{\text{new},k}^+ \Big) + 4 \frac{b^2(1-b^{2\alpha})}{1-b^2} r \gamma_*^2 \zeta_*^+ \phi^+ \frac{\kappa_s^+}{\sqrt{1-(\zeta_*^+)^2}} + \frac{c\zeta\lambda^-}{24} \Big| X_{j,k-1} \Big) \geq 1 - \max\{p_d(\alpha, \zeta), p_e(\alpha, \zeta)\} \quad (85)$$

4) $\|B_k\|$: Consider $\|B_k\|_2$. Let $Z_t := E_{\text{new},\perp}' \Phi_0(L_t - e_t)(L_t' - e_t') \Phi_0 E_{\text{new}}$ which is of size $(n-c) \times c$. Then $B_k = \frac{1}{\alpha} \sum_t Z_t$. As $E_{\text{new},\perp}' \Phi_0 P_{\text{new}} a_{t,\text{new}} = E_{\text{new},\perp}' E_{\text{new}} R_{\text{new}} a_{t,\text{new}} = 0$, $E_{\text{new},\perp}' \Phi_0(L_t - e_t) = E_{\text{new},\perp}'(D_* a_{t,*} - \Phi_0 e_t)$, $E_{\text{new}}' \Phi_0(L_t - e_t) = R_{\text{new}} a_{t,\text{new}} + E_{\text{new}}' D_* a_{t,*} - (R_{\text{new}}')^{-1} D_{\text{new}}' e_t$. Thus,

$$\begin{aligned} Z_t &= E_{\text{new},\perp}' (D_* a_{t,*} - \Phi_0 e_t) (a_{t,\text{new}}' R_{\text{new}}' + a_{t,*}' D_*' E_{\text{new}} - e_t' D_{\text{new}} (R_{\text{new}})^{-1}) \\ &= E_{\text{new},\perp}' (D_* a_{t,*} - \Phi_0 I_{T_t} [(\Phi_{k-1})_{T_t}' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' [(\Phi_{k-1} P_*) a_{t,*} + D_{\text{new},k-1} a_{t,\text{new}}]) (a_{t,\text{new}}' R_{\text{new}}' + a_{t,*}' D_*' E_{\text{new}} - [(\Phi_{k-1} P_*) a_{t,*} \\ &\quad + D_{\text{new},k-1} a_{t,\text{new}}]' I_{T_t} [(\Phi_{k-1})_{T_t}' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' D_{\text{new}} (R_{\text{new}})^{-1}) \\ &= E_{\text{new},\perp}' (D_* - \Phi_0 I_{T_t} [(\Phi_{k-1})_{T_t}' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' (\Phi_{k-1} P_*) a_{t,*} a_{t,\text{new}}' (D_*' E_{\text{new}} - P_*' \Phi_{k-1} I_{T_t} [(\Phi_{k-1})_{T_t}' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' D_{\text{new}} (R_{\text{new}})^{-1}) \\ &\quad + E_{\text{new},\perp}' (D_* - \Phi_0 I_{T_t} [(\Phi_{k-1})_{T_t}' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' (\Phi_{k-1} P_*) a_{t,*} a_{t,\text{new}}' (R_{\text{new}}' - D_{\text{new},k-1}' I_{T_t} [(\Phi_{k-1})_{T_t}' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' D_{\text{new}} (R_{\text{new}})^{-1}) \\ &\quad + E_{\text{new},\perp}' (-\Phi_0 I_{T_t} [(\Phi_{k-1})_{T_t}' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' D_{\text{new},k-1} a_{t,\text{new}} a_{t,*}' (D_*' E_{\text{new}} - P_*' \Phi_{k-1} I_{T_t} [(\Phi_{k-1})_{T_t}' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' D_{\text{new}} (R_{\text{new}})^{-1}) \\ &\quad + E_{\text{new},\perp}' (-\Phi_0 I_{T_t} [(\Phi_{k-1})_{T_t}' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' D_{\text{new},k-1} a_{t,\text{new}} a_{t,\text{new}}' (R_{\text{new}}' - D_{\text{new},k-1}' I_{T_t} [(\Phi_{k-1})_{T_t}' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' D_{\text{new}} (R_{\text{new}})^{-1}) \end{aligned} \quad (86)$$

(a) Using (22) and similar procedure to get (54), we have

$$\begin{aligned} \mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{t \in t_{j,k}} E_{\text{new},\perp}' (D_* - \Phi_0 I_{T_t} [(\Phi_{k-1})_{T_t}' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' (\Phi_{k-1} P_*) a_{t,*} a_{t,\text{new}}' (D_*' E_{\text{new}} - P_*' \Phi_{k-1} I_{T_t} [(\Phi_{k-1})_{T_t}' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' D_{\text{new}} (R_{\text{new}})^{-1}) \right\|_2 \right. \\ \left. \leq (\zeta_*^+ + \phi^+ \zeta_*^+) (\zeta_*^+ + \zeta_*^+ \phi^+ \frac{\kappa_s^+}{\sqrt{1-(\zeta_*^+)^2}}) (\lambda^+ + \frac{b^2(1-b^{2\alpha})}{1-b^2} r \gamma_*^2) + \frac{c\zeta\lambda^-}{96} \Big| X_{j,k-1} \right) \geq 1 - p_{f_1}(\alpha, \zeta) \end{aligned} \quad (87)$$

where

$$\begin{aligned} p_{f_1}(\alpha, \zeta) &= nF \left(\alpha, \frac{c\zeta\lambda^-}{384}, \frac{(1-b^{2\alpha})(1-b)}{1+b} r \gamma_*^2 (\zeta_*^+)^2 \left(1 + \phi^+ \left(1 + \phi^+ \frac{\kappa_s^+}{\sqrt{1-(\zeta_*^+)^2}} \right) \right) \right) + nF \left(\alpha, \frac{c\zeta\lambda^-}{384}, \right. \\ &\quad \left. 2 \frac{b(1-b)(1-b^{2(\alpha-1)})}{1+b} r \gamma_*^2 (\zeta_*^+)^2 (1 + \phi^+) \left(1 + \phi^+ \frac{\kappa_s^+}{\sqrt{1-(\zeta_*^+)^2}} \right) \right) + nF \left(\alpha, \frac{c\zeta\lambda^-}{384}, 2 \frac{(1-b^\alpha)^2}{1+b} r \gamma_*^2 (\zeta_*^+)^2 \right. \\ &\quad \left. (1 + \phi^+) \left(1 + \phi^+ \frac{\kappa_s^+}{\sqrt{1-(\zeta_*^+)^2}} \right) \right) + nF \left(\alpha, \frac{c\zeta\lambda^-}{384}, 2 \frac{b(1-b^{2\alpha})}{1+b} r \gamma_*^2 (\zeta_*^+)^2 (1 + \phi^+) \left(1 + \phi^+ \frac{\kappa_s^+}{\sqrt{1-(\zeta_*^+)^2}} \right) \right) \end{aligned} \quad (88)$$

(b) Using (22) and similar procedure to get (43), we have

$$\begin{aligned} \mathbf{P} \left(\left\| E_{\text{new},\perp}' (D_* - \Phi_0 I_{T_t} [(\Phi_{k-1})_{T_t}' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' (\Phi_{k-1} P_*) a_{t,*} a_{t,\text{new}}' (R_{\text{new}}' - D_{\text{new},k-1}' I_{T_t} [(\Phi_{k-1})_{T_t}' (\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' D_{\text{new}} (R_{\text{new}})^{-1}) \right\|_2 \right. \\ \left. \leq \frac{b^2(1-b^{2\alpha})}{1-b^2} \sqrt{cr} \gamma_* \gamma_{\text{new},k} (\zeta_*^+ + \phi^+ \zeta_*^+) \left(1 + (\kappa_s^+)^2 \zeta_{k-1}^+ \frac{\phi^+}{\sqrt{1-(\zeta_*^+)^2}} \right) + \frac{c\zeta\lambda^-}{96} \Big| X_{j,k-1} \right) \geq 1 - p_{f_2}(\alpha, \zeta) \end{aligned} \quad (89)$$

where

$$\begin{aligned} p_{f_2}(\alpha, \zeta) &= nF \left(\alpha, \frac{c\zeta\lambda^-}{192}, 2 \frac{1+b-2\sqrt{b^{\alpha+1}(1+b-b^{\alpha+1})}}{1+b} \sqrt{cr} \gamma_* \gamma_{\text{new},k} (\zeta_*^+ + \phi^+ \zeta_*^+) (1 + (\kappa_s^+)^2 \zeta_{k-1}^+ \frac{\phi^+}{\sqrt{1-(\zeta_*^+)^2}}) \right) + \\ &\quad nF \left(\alpha, \frac{c\zeta\lambda^-}{192}, 4 \frac{b(1-b^{2\alpha})}{1+b} \sqrt{cr} \gamma_* \gamma_{\text{new},k} (\zeta_*^+ + \phi^+ \zeta_*^+) \left(1 + (\kappa_s^+)^2 \zeta_{k-1}^+ \frac{\phi^+}{\sqrt{1-(\zeta_*^+)^2}} \right) \right) \end{aligned}$$

and

$$\mathbf{P} \left(\|E_{\text{new},\perp}'(-\Phi_0 I_{T_t}[(\Phi_{k-1})_{T_t}'(\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' D_{\text{new},k-1}) a_{t,\text{new}} a_{t,*}' (D_{\text{new}}' E_{\text{new}} - P_{\text{new}}' \Phi_{k-1} I_{T_t}[(\Phi_{k-1})_{T_t}'(\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' D_{\text{new}} (R_{\text{new}})^{-1}) \|_2 \leq \frac{b^2(1-b^{2\alpha})}{1-b^2} \sqrt{c r \gamma_* \gamma_{\text{new},k}} \phi^+ \kappa_s^+ \zeta_{k-1}^+ \left(\zeta_*^+ + \zeta_*^+ \phi^+ \frac{\kappa_s^+}{\sqrt{1-(\zeta_*^+)^2}} \right) + \frac{c \zeta \lambda^-}{96} |X_{j,k-1}| \right) \geq 1 - p_{f_3}(\alpha, \zeta) \quad (90)$$

where

$$p_{f_3}(\alpha, \zeta) = nF \left(\alpha, \frac{c \zeta \lambda^-}{192}, 2 \frac{1+b-2\sqrt{b^{\alpha+1}(1+b-b^{\alpha+1})}}{1+b} \sqrt{c r \gamma_* \gamma_{\text{new},k}} \phi^+ \kappa_s^+ \zeta_{k-1}^+ \left(\zeta_*^+ + \zeta_*^+ \phi^+ \frac{\kappa_s^+}{\sqrt{1-(\zeta_*^+)^2}} \right) \right) + nF \left(\alpha, \frac{c \zeta \lambda^-}{192}, 4 \frac{b(1-b^{2\alpha})}{1+b} \sqrt{c r \gamma_* \gamma_{\text{new},k}} \phi^+ \kappa_s^+ \zeta_{k-1}^+ \left(\zeta_*^+ + \zeta_*^+ \phi^+ \frac{\kappa_s^+}{\sqrt{1-(\zeta_*^+)^2}} \right) \right)$$

(c) Using (22) and similar procedure to get (34), we have

$$\mathbf{P} \left(\left\| \frac{1}{\alpha} \sum_{t \in t_{j,k}} E_{\text{new},\perp}'(-\Phi_0 I_{T_t}[(\Phi_{k-1})_{T_t}'(\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' D_{\text{new},k-1}) a_{t,\text{new}} a_{t,\text{new}}' (R_{\text{new}}' - D_{\text{new},k-1}' I_{T_t}[(\Phi_{k-1})_{T_t}'(\Phi_{k-1})_{T_t}]^{-1} I_{T_t}' D_{\text{new}} (R_{\text{new}})^{-1}) \right\|_2 \leq \zeta_{k-1}^+ \phi^+ \kappa_s^+ \left(1 + \phi^+ \zeta_{k-1}^+ \frac{(\kappa_s^+)^2}{\sqrt{1-(\zeta_*^+)^2}} \right) \left(\lambda_{\text{new},k}^+ + \frac{b^2(1-b^{2\alpha})}{1-b^2} c \gamma_{\text{new},k-1}^2 \right) + \frac{c \zeta \lambda^-}{96} |X_{j,k-1}| \right) \geq 1 - p_{f_4}(\alpha, \zeta) \quad (91)$$

where

$$p_{f_4}(\alpha, \zeta) = nF \left(\alpha, \frac{c \zeta \lambda^-}{384}, \frac{(1-b^{2\alpha})(1-b)}{1+b} c \gamma_{\text{new},k}^2 \zeta_{k-1}^+ \phi^+ \kappa_s^+ \left(1 + \phi^+ \zeta_{k-1}^+ \frac{(\kappa_s^+)^2}{\sqrt{1-(\zeta_*^+)^2}} \right) \right) + nF \left(\alpha, \frac{c \zeta \lambda^-}{384}, 2 \frac{b(1-b)(1-b^{2(\alpha-1)})}{1+b} c \gamma_{\text{new},k}^2 (\zeta_{k-1}^+ \phi^+)^2 \frac{(\kappa_s^+)^3}{\sqrt{1-(\zeta_*^+)^2}} \right) + nF \left(\alpha, \frac{c \zeta \lambda^-}{384}, 2 \frac{(1-b^\alpha)^2}{1+b} c \gamma_{\text{new},k}^2 \zeta_{k-1}^+ \phi^+ \kappa_s^+ \left(1 + \phi^+ \zeta_{k-1}^+ \frac{(\kappa_s^+)^2}{\sqrt{1-(\zeta_*^+)^2}} \right) \right) + nF \left(\alpha, \frac{c \zeta \lambda^-}{384}, 4 \frac{b(1-b^{2\alpha})}{1+b} c \gamma_{\text{new},k} \gamma_{\text{new},k-1} \zeta_{k-1}^+ \phi^+ \kappa_s^+ \left(1 + \phi^+ \zeta_{k-1}^+ \frac{(\kappa_s^+)^2}{\sqrt{1-(\zeta_*^+)^2}} \right) \right) \quad (92)$$

Using (87), (89), (90) and (91) and the union bound, for any $X_{j,k-1} \in \Gamma_{j,k-1}$,

$$\mathbf{P} \left(\|B_k\|_2 \leq (\zeta_*^+ + \phi^+ \zeta_*^+) \left(\zeta_*^+ + \zeta_*^+ \phi^+ \frac{\kappa_s^+}{\sqrt{1-(\zeta_*^+)^2}} \right) \left(\lambda^+ + \frac{b^2(1-b^{2\alpha})}{1-b^2} r \gamma_*^2 \right) + \zeta_{k-1}^+ \phi^+ \kappa_s^+ \left(1 + \phi^+ \zeta_{k-1}^+ \frac{(\kappa_s^+)^2}{\sqrt{1-(\zeta_*^+)^2}} \right) \left(\lambda_{\text{new},k}^+ + \frac{b^2(1-b^{2\alpha})}{1-b^2} c \gamma_{\text{new},k-1}^2 \right) + \frac{b^2(1-b^{2\alpha})}{1-b^2} \sqrt{c r \gamma_* \gamma_{\text{new},k}} \left((\zeta_*^+ + \phi^+ \zeta_*^+) \left(1 + (\kappa_s^+)^2 \zeta_{k-1}^+ \frac{\phi^+}{\sqrt{1-(\zeta_*^+)^2}} \right) + \phi^+ \kappa_s^+ \zeta_{k-1}^+ \left(\zeta_*^+ + \zeta_*^+ \phi^+ \frac{\kappa_s^+}{\sqrt{1-(\zeta_*^+)^2}} \right) \right) + \frac{c \zeta \lambda^-}{24} |X_{j,k-1}| \right) \geq 1 - p_f(\alpha, \zeta) \quad (93)$$

where $p_f(\alpha, \zeta) = p_{f_1}(\alpha, \zeta) + p_{f_2}(\alpha, \zeta) + p_{f_3}(\alpha, \zeta) + p_{f_4}(\alpha, \zeta)$.

Using (70), (85) and (93) and the union bound, for any $X_{j,k-1} \in \Gamma_{j,k-1}$,

$$\mathbf{P} \left(\|\mathcal{H}_k\|_2 \leq b_{\mathcal{H}} + \frac{c \zeta \lambda^-}{8} |X_{j,k-1}| \right) \geq 1 - p_c(\alpha, \zeta) - p_f(\alpha, \zeta) - \max\{p_d(\alpha, \zeta), p_e(\alpha, \zeta)\} \quad (94)$$

where

$$\begin{aligned}
b_{\mathcal{H}} := & (\phi^+)^2(\zeta_*^+)^2\lambda^+ + (\phi^+)^2(\kappa_s^+\zeta_{k-1}^+)^2\lambda_{\text{new},k}^+ + 2\frac{b^2(1-b^{2\alpha})}{1-b^2}\sqrt{cr}\gamma_*\gamma_{\text{new},k}\zeta_*^+\zeta_{k-1}^+\kappa_s^+(\phi^+)^2 + \frac{b^2(1-b^{2\alpha})}{1-b^2}r\gamma_*^2(\phi^+\zeta_*^+)^2 + \\
& \frac{b^2(1-b^{2\alpha})}{1-b^2}c\gamma_{\text{new},k-1}^2(\phi^+\kappa_s^+\zeta_{k-1}^+)^2 + 2\phi^+\kappa_s^+\frac{(\zeta_*^+)^2}{\sqrt{1-(\zeta_*^+)^2}}\left(\lambda^+ + \frac{b^2(1-b^{2\alpha})}{1-b^2}r\gamma_*^2\right) + \\
& 2\phi^+\max\{0.15c\zeta, \zeta_{k-1}^+\}\frac{(\kappa_s^+)^2}{\sqrt{1-(\zeta_*^+)^2}}\left(\lambda_{\text{new},k}^+ + \frac{b^2(1-b^{2\alpha})}{1-b^2}\eta\lambda_{\text{new},k}^+\right) + 4\frac{b^2(1-b^{2\alpha})}{1-b^2}r\gamma_*^2\zeta_*^+\phi^+\frac{\kappa_s^+}{\sqrt{1-(\zeta_*^+)^2}} + \\
& (\zeta_*^+ + \phi^+\zeta_*^+)\left(\zeta_*^+ + \zeta_*^+\phi^+\frac{\kappa_s^+}{\sqrt{1-(\zeta_*^+)^2}}\right)\left(\lambda^+ + \frac{b^2(1-b^{2\alpha})}{1-b^2}r\gamma_*^2\right) + \zeta_{k-1}^+\phi^+\kappa_s^+\left(1 + \phi^+\zeta_{k-1}^+\frac{(\kappa_s^+)^2}{\sqrt{1-(\zeta_*^+)^2}}\right) \\
& \left(\lambda_{\text{new},k}^+ + \frac{b^2(1-b^{2\alpha})}{1-b^2}c\gamma_{\text{new},k-1}^2\right) + \frac{b^2(1-b^{2\alpha})}{1-b^2}\sqrt{cr}\gamma_*\gamma_{\text{new},k}\left((\zeta_*^+ + \phi^+\zeta_*^+)\left(1 + (\kappa_s^+)^2\zeta_{k-1}^+\frac{\phi^+}{\sqrt{1-(\zeta_*^+)^2}}\right)\right. \\
& \left. + \phi^+\kappa_s^+\zeta_{k-1}^+\left(\zeta_*^+ + \zeta_*^+\phi^+\frac{\kappa_s^+}{\sqrt{1-(\zeta_*^+)^2}}\right)\right) \tag{95}
\end{aligned}$$

Remark 10.5: As shown in Remark 10.4, when $k = 1$, there is no $a_{t_j+(k-1)\alpha-1,\text{new}}$, leading to changes in bounds $b_{A_k}, b_{A_{k,\perp}}, b_{\mathcal{H}}$, in which case ζ_k^+ decreases exponentially with same parameters given above with larger b . ■

APPENDIX

A. Proof of Lemma 2.10

Proof: Because P, Q and \hat{P} are basis matrix, $P'P = I, Q'Q = I$ and $\hat{P}'\hat{P} = I$.

- 1) Using $P'P = I$ and $\|M\|_2^2 = \|MM'\|_2$, $\|(I - \hat{P}\hat{P}')PP'\|_2 = \|(I - \hat{P}\hat{P}')P\|_2$. Similarly, $\|(I - PP')\hat{P}\hat{P}'\|_2 = \|(I - PP')\hat{P}\|_2$. Let $D_1 = (I - \hat{P}\hat{P}')PP'$ and let $D_2 = (I - PP')\hat{P}\hat{P}'$. Notice that $\|D_1\|_2 = \sqrt{\lambda_{\max}(D_1'D_1)} = \sqrt{\|D_1'D_1\|_2}$ and $\|D_2\|_2 = \sqrt{\lambda_{\max}(D_2'D_2)} = \sqrt{\|D_2'D_2\|_2}$. So, in order to show $\|D_1\|_2 = \|D_2\|_2$, it suffices to show that $\|D_1'D_1\|_2 = \|D_2'D_2\|_2$. Let $P'\hat{P} \stackrel{SVD}{=} U\Sigma V'$. Then, $D_1'D_1 = P(I - P'\hat{P}\hat{P}'P)P' = PU(I - \Sigma^2)U'P'$ and $D_2'D_2 = \hat{P}(I - \hat{P}PP'\hat{P})\hat{P}' = \hat{P}V(I - \Sigma^2)V'\hat{P}'$ are the compact SVD's of $D_1'D_1$ and $D_2'D_2$ respectively. Therefore, $\|D_1'D_1\|_2 = \|D_2'D_2\|_2 = \|I - \Sigma^2\|_2$ and hence $\|(I - \hat{P}\hat{P}')PP'\|_2 = \|(I - PP')\hat{P}\hat{P}'\|_2$.
- 2) $\|PP' - \hat{P}\hat{P}'\|_2 = \|PP' - \hat{P}\hat{P}'PP' + \hat{P}\hat{P}'PP' - \hat{P}\hat{P}'\|_2 \leq \|(I - \hat{P}\hat{P}')PP'\|_2 + \|(I - PP')\hat{P}\hat{P}'\|_2 = 2\zeta_*$.
- 3) Since $Q'P = 0$, then $\|Q'\hat{P}\|_2 = \|Q'(I - PP')\hat{P}\|_2 \leq \|(I - PP')\hat{P}\|_2 = \zeta_*$.
- 4) Let $M = (I - \hat{P}\hat{P}')Q$. Then $M'M = Q'(I - \hat{P}\hat{P}')Q$ and so $\sigma_i((I - \hat{P}\hat{P}')Q) = \sqrt{\lambda_i(Q'(I - \hat{P}\hat{P}')Q)}$. Clearly, $\lambda_{\max}(Q'(I - \hat{P}\hat{P}')Q) \leq 1$. By Weyl's Theorem, $\lambda_{\min}(Q'(I - \hat{P}\hat{P}')Q) \geq 1 - \lambda_{\max}(Q'\hat{P}\hat{P}'Q) = 1 - \|Q'\hat{P}\|_2^2 \geq 1 - \zeta_*^2$. Therefore, $\sqrt{1 - \zeta_*^2} \leq \sigma_i((I - \hat{P}\hat{P}')Q) \leq 1$. ■

B. Proof of Lemma 2.11

Proof: It is easy to see that $\mathbf{P}(\mathcal{B}^e, \mathcal{C}^e) = \mathbf{E}[\mathbb{I}_{\mathcal{B}}(X, Y)\mathbb{I}_{\mathcal{C}}(X)]$. If $\mathbf{E}[\mathbb{I}_{\mathcal{B}}(X, Y)|X] \geq p$ for all $X \in \mathcal{C}$, this means that $\mathbf{E}[\mathbb{I}_{\mathcal{B}}(X, Y)|X]\mathbb{I}_{\mathcal{C}}(X) \geq p\mathbb{I}_{\mathcal{C}}(X)$. This, in turn, implies that

$$\mathbf{P}(\mathcal{B}^e, \mathcal{C}^e) = \mathbf{E}[\mathbb{I}_{\mathcal{B}}(X, Y)\mathbb{I}_{\mathcal{C}}(X)] = \mathbf{E}[\mathbf{E}[\mathbb{I}_{\mathcal{B}}(X, Y)|X]\mathbb{I}_{\mathcal{C}}(X)] \geq p\mathbf{E}[\mathbb{I}_{\mathcal{C}}(X)].$$

Recall from Definition 2.4 that $\mathbf{P}(\mathcal{B}^e|X) = \mathbf{E}[\mathbb{I}_{\mathcal{B}}(X, Y)|X]$ and $\mathbf{P}(\mathcal{C}^e) = \mathbf{E}[\mathbb{I}_{\mathcal{C}}(X)]$. Thus, we conclude that if $\mathbf{P}(\mathcal{B}^e|X) \geq p$ for all $X \in \mathcal{C}$, then $\mathbf{P}(\mathcal{B}^e, \mathcal{C}^e) \geq p\mathbf{P}(\mathcal{C}^e)$. Using the definition of $\mathbf{P}(\mathcal{B}^e|\mathcal{C}^e)$, the claim follows. ■

C. Proof of Theorem 2.13

Proof of Theorem: The matrix Laplace transform method, Proposition 3.1, states that

$$\mathbf{P} \left(\lambda_{\max} \left(\sum_k \mathbf{X}_k \right) \geq t \right) \leq \inf_{\theta > 0} \left\{ e^{-\theta t} \cdot \mathbf{E} \text{tr exp} \left(\sum_k \theta \mathbf{X}_k \right) \right\}. \quad (96)$$

The main difficulty in the proof is to bound the matrix mgf, which we accomplish by an iterative argument that alternates between symmetrization and cumulant bounds.

Let us detail the first step of the iteration. Denote conditional expectation $\mathbf{E}(XY|g(Y))$ as $\mathbf{E}_{X,Y|g(Y)}(XY)$, then

$$\begin{aligned} \mathbf{E} \text{tr exp} \left(\sum_k \theta \mathbf{X}_k \right) &= \mathbf{E}_{\mathbf{X}_1, \dots, \mathbf{X}_{n-1}} \mathbf{E}_{\mathbf{X}_n | \mathbf{X}_1, \dots, \mathbf{X}_{n-1}} \left[\text{tr exp} \left(\sum_{k=1}^{n-1} \theta \mathbf{X}_k + \theta \mathbf{X}_n \right) \right] \\ &\leq \mathbf{E}_{\mathbf{X}_1, \dots, \mathbf{X}_{n-1}} \mathbf{E}_{\epsilon, \mathbf{X}_n | \mathbf{X}_1, \dots, \mathbf{X}_{n-1}} \left[\text{tr exp} \left(\sum_{k=1}^{n-1} \theta \mathbf{X}_k + 2\epsilon \theta \mathbf{X}_n \right) \right] \\ &= \mathbf{E}_{\mathbf{X}_1, \dots, \mathbf{X}_n} \mathbf{E}_{\epsilon | \mathbf{X}_1, \dots, \mathbf{X}_n} \left[\text{tr exp} \left(\sum_{k=1}^{n-1} \theta \mathbf{X}_k + 2\epsilon \theta \mathbf{X}_n \right) \right] \\ &\leq \mathbf{E}_{\mathbf{X}_1, \dots, \mathbf{X}_n} \text{tr exp} \left(\sum_{k=1}^{n-1} \theta \mathbf{X}_k + \log \mathbf{E}_{\epsilon | \mathbf{X}_1, \dots, \mathbf{X}_n} [e^{2\epsilon \theta \mathbf{X}_n}] \right) \\ &\leq \mathbf{E}_{\mathbf{X}_1, \dots, \mathbf{X}_n} \text{tr exp} \left(\sum_{k=1}^{n-1} \theta \mathbf{X}_k + 2\theta^2 \mathbf{A}_n^2 \right). \end{aligned}$$

The first identity is the tower property of conditional expectation. In the second line, we invoke the symmetrization method, Lemma 7.6, conditional on $\mathbf{X}_1, \dots, \mathbf{X}_{n-1}$, and then we relax the conditioning on the inner expectation to $\mathbf{X}_1, \dots, \mathbf{X}_n$. By construction, the Rademacher variable ϵ is independent from $\mathbf{X}_1, \dots, \mathbf{X}_n$, so we can apply the concavity result, Corollary 3.3, conditional on $\mathbf{X}_1, \dots, \mathbf{X}_n$. Finally, we use the fact (2.5) that the trace exponential is monotone to introduce the Azuma cgf bound, Lemma 7.7, in the last inequality.

By iteration, we achieve

$$\mathbf{E} \text{tr exp} \left(\sum_k \theta \mathbf{X}_k \right) \leq \text{tr exp} \left(2\theta^2 \sum_k \mathbf{A}_k^2 \right). \quad (97)$$

Note that this procedure relies on the fact that the sequence $\{\mathbf{A}_k\}$ of upper bounds does not depend on the values of the random sequence $\{\mathbf{X}_k\}$. Substitute the mgf bound (97) into the Laplace transform bound (96), and observe that the infimum is achieved when $\theta = t/4\sigma^2$. ■

D. Proof of Corollary 2.14

Proof: Define the dilation of an $n_1 \times n_2$ matrix M as $\text{dilation}(M) := \begin{bmatrix} 0 & M' \\ M & 0 \end{bmatrix}$. Notice that this is an $(n_1 + n_2) \times (n_1 + n_2)$ Hermitian matrix [26]. As shown in [26, equation 2.12],

$$\lambda_{\max}(\text{dilation}(M)) = \|\text{dilation}(M)\|_2 = \|M\|_2 \quad (98)$$

Thus, the corollary assumptions imply that $\mathbf{P}(\|\text{dilation}(Z_t)\|_2 \leq b_1 | X) = 1$ for all $X \in \mathcal{C}$. Using (98), the corollary assumptions also imply that $\mathbf{E}_{t-1}(\text{dilation}(Z_t) | X) = \text{dilation}(\mathbf{E}_{t-1}(Z_t | X)) = 0$ for all $X \in \mathcal{C}$. Thus, applying Corollary 2.13 for the sequence $\{\text{dilation}(Z_t)\}$, we get that,

$$\mathbf{P} \left(\lambda_{\max} \left(\frac{1}{\alpha} \sum_{t=1}^{\alpha} \text{dilation}(Z_t) \right) \leq \epsilon | X \right) \geq 1 - (n_1 + n_2) \exp \left(-\frac{\alpha \epsilon^2}{32b_1^2} \right) \text{ for all } X \in \mathcal{C}$$

Using (98), $\lambda_{\max}(\frac{1}{\alpha} \sum_{t=1}^{\alpha} \text{dilation}(Z_t)) = \lambda_{\max}(\text{dilation}(\frac{1}{\alpha} \sum_{t=1}^{\alpha} Z_t)) = \|\frac{1}{\alpha} \sum_{t=1}^{\alpha} Z_t\|_2$ and this gives the final result. ■

E. Proof of Corollary 2.15

Proof:

- 1) Since, for any $X \in \mathcal{C}$, conditioned on X , the Z_t 's are adapted, the same is also true for $Z_t - g(X, Z_{1:t-1})$ for any function of X and $Z_{1:t-1}$. Let $Y_t := Z_t - \mathbf{E}_{t-1}(Z_t | X)$. Thus, for any $X \in \mathcal{C}$, conditioned on X , the Y_t 's are adapted. Also,

clearly $\mathbf{E}_{t-1}(Y_t|X) = 0$. Since for all $X \in \mathcal{C}$, $\mathbf{P}(b_1 I \preceq Z_t \preceq b_2 I|X) = 1$ and since $\lambda_{\max}(\cdot)$ is a convex function, and $\lambda_{\min}(\cdot)$ is a concave function, of a Hermitian matrix, thus $b_1 I \preceq \mathbf{E}_{t-1}(Z_t|X) \preceq b_2 I$ w.p. one for all $X \in \mathcal{C}$. Therefore, $\mathbf{P}(Y_t^2 \preceq (b_2 - b_1)^2 I|X) = 1$ for all $X \in \mathcal{C}$. Thus, for Theorem 2.13, $\sigma^2 = \|\sum_{t=1}^{\alpha} (b_2 - b_1)^2 I\|_2 = \alpha(b_2 - b_1)^2$. For any $X \in \mathcal{C}$, applying Theorem 2.13 for $\{Y_t\}$'s conditioned on X , we get that, for any $\epsilon > 0$,

$$\mathbf{P}\left(\lambda_{\max}\left(\frac{1}{\alpha}\sum_{t=1}^{\alpha} Y_t\right) \leq \epsilon|X\right) > 1 - n \exp\left(-\frac{\alpha\epsilon^2}{8(b_2 - b_1)^2}\right) \text{ for all } X \in \mathcal{C}$$

By Weyl's theorem, $\lambda_{\max}(\frac{1}{\alpha}\sum_{t=1}^{\alpha} Y_t) = \lambda_{\max}(\frac{1}{\alpha}\sum_{t=1}^{\alpha} (Z_t - \mathbf{E}_{t-1}(Z_t|X))) \geq \lambda_{\max}(\frac{1}{\alpha}\sum_{t=1}^{\alpha} Z_t) + \lambda_{\min}(\frac{1}{\alpha}\sum_{t=1}^{\alpha} -\mathbf{E}_{t-1}(Z_t|X))$. Since $\lambda_{\min}(\frac{1}{\alpha}\sum_{t=1}^{\alpha} -\mathbf{E}_{t-1}(Z_t|X)) = -\lambda_{\max}(\frac{1}{\alpha}\sum_{t=1}^{\alpha} \mathbf{E}_{t-1}(Z_t|X)) \geq -b_4$, thus $\lambda_{\max}(\frac{1}{\alpha}\sum_{t=1}^{\alpha} Y_t) \geq \lambda_{\max}(\frac{1}{\alpha}\sum_{t=1}^{\alpha} Z_t) - b_4$. Therefore,

$$\mathbf{P}\left(\lambda_{\max}\left(\frac{1}{\alpha}\sum_{t=1}^{\alpha} Z_t\right) \leq b_4 + \epsilon|X\right) > 1 - n \exp\left(-\frac{\alpha\epsilon^2}{8(b_2 - b_1)^2}\right) \text{ for all } X \in \mathcal{C}$$

2) Let $Y_t = \mathbf{E}_{t-1}(Z_t|X) - Z_t$. As before, $\mathbf{E}_{t-1}(Y_t|X) = 0$ and conditioned on any $X \in \mathcal{C}$, the Y_t 's are independent and $\mathbf{P}(Y_t^2 \preceq (b_2 - b_1)^2 I|X) = 1$. As before, applying Theorem 2.13, we get that for any $\epsilon > 0$,

$$\mathbf{P}\left(\lambda_{\max}\left(\frac{1}{\alpha}\sum_{t=1}^{\alpha} Y_t\right) \leq \epsilon|X\right) > 1 - n \exp\left(-\frac{\alpha\epsilon^2}{8(b_2 - b_1)^2}\right) \text{ for all } X \in \mathcal{C}$$

By Weyl's theorem, $\lambda_{\max}(\frac{1}{\alpha}\sum_{t=1}^{\alpha} Y_t) = \lambda_{\max}(\frac{1}{\alpha}\sum_{t=1}^{\alpha} (\mathbf{E}_{t-1}(Z_t|X) - Z_t)) \geq \lambda_{\min}(\frac{1}{\alpha}\sum_{t=1}^{\alpha} \mathbf{E}_{t-1}(Z_t|X)) + \lambda_{\max}(\frac{1}{\alpha}\sum_{t=1}^{\alpha} -Z_t) = \lambda_{\min}(\frac{1}{\alpha}\sum_{t=1}^{\alpha} \mathbf{E}_{t-1}(Z_t|X)) - \lambda_{\min}(\frac{1}{\alpha}\sum_{t=1}^{\alpha} Z_t) \geq b_3 - \lambda_{\min}(\frac{1}{\alpha}\sum_{t=1}^{\alpha} Z_t)$. Therefore, for any $\epsilon > 0$,

$$\mathbf{P}\left(\lambda_{\min}\left(\frac{1}{\alpha}\sum_{t=1}^{\alpha} Z_t\right) \geq b_3 - \epsilon|X\right) \geq 1 - n \exp\left(-\frac{\alpha\epsilon^2}{8(b_2 - b_1)^2}\right) \text{ for all } X \in \mathcal{C}$$

■

F. Proof of Lemma 2.17

Proof: Denote conditional expectation $\mathbf{E}(Xh(Y)|g(Y))$ as $\mathbf{E}_{X,Y|g(Y)}(Xh(Y))$, then

$$\begin{aligned} \mathbf{E}(Xh(Y)|g(Y)) &= \mathbf{E}_{Y|g(Y)}(\mathbf{E}_{X|Y,g(Y)}(Xh(Y))) \\ &= \mathbf{E}_{Y|g(Y)}(\mathbf{E}_{X|Y}(Xh(Y))) \\ &= \mathbf{E}_{Y|g(Y)}(\mathbf{E}_{X|Y}(X)h(Y)) \\ &= \mathbf{E}_{Y|g(Y)}(\mathbf{E}(X)h(Y)) \\ &= \mathbf{E}(X)\mathbf{E}(h(Y)|g(Y)). \end{aligned} \tag{99}$$

■

G. Proof of Lemma 3.7

Proof: Let $A = I - PP'$. By definition, $\delta_s(A) := \max\{\max_{|T| \leq s}(\lambda_{\max}(A'_T A_T) - 1), \max_{|T| \leq s}(1 - \lambda_{\min}(A'_T A_T))\}$. Notice that $A'_T A_T = I - I'_T P P' I_T$. Since $I'_T P P' I_T$ is p.s.d., by Weyl's theorem, $\lambda_{\max}(A'_T A_T) \leq 1$. Since $\lambda_{\max}(A'_T A_T) - 1 \leq 0$ while $1 - \lambda_{\min}(A'_T A_T) \geq 0$, thus,

$$\delta_s(I - PP') = \max_{|T| \leq s} \left(1 - \lambda_{\min}(I - I'_T P P' I_T)\right) \tag{100}$$

By Definition, $\kappa_s(P) = \max_{|T| \leq s} \frac{\|I'_T P\|_2}{\|P\|_2} = \max_{|T| \leq s} \|I'_T P\|_2$. Notice that $\|I'_T P\|_2^2 = \lambda_{\max}(I'_T P P' I_T) = 1 - \lambda_{\min}(I - I'_T P P' I_T)$, and so

$$\kappa_s^2(P) = \max_{|T| \leq s} \left(1 - \lambda_{\min}(I - I'_T P P' I_T)\right) \tag{101}$$

²This follows because $B = I'_T P P' I_T$ is a Hermitian matrix. Let $B = U \Sigma U'$ be its EVD. Since $U U' = I$, $\lambda_{\min}(I - B) = \lambda_{\min}(U(I - \Sigma)U') = \lambda_{\min}(I - \Sigma) = 1 - \lambda_{\max}(\Sigma) = 1 - \lambda_{\max}(B)$.

From (100) and (101), we get $\delta_s(I - PP') = \kappa_s^2(P)$. ■

REFERENCES

- [1] D. Skocaj and A. Leonardis, “Weighted and robust incremental method for subspace learning,” in *IEEE Intl. Conf. on Computer Vision (ICCV)*, vol. 2, Oct 2003, pp. 1494–1501.
- [2] Y. Li, L. Xu, J. Morphet, and R. Jacobs, “An integrated algorithm of incremental and robust pca,” in *IEEE Intl. Conf. Image Proc. (ICIP)*, 2003, pp. 245–248.
- [3] F. D. L. Torre and M. J. Black, “A framework for robust subspace learning,” *International Journal of Computer Vision*, vol. 54, pp. 117–142, 2003.
- [4] E. J. Candès, X. Li, Y. Ma, and J. Wright, “Robust principal component analysis?” *Journal of ACM*, vol. 58, no. 3, 2011.
- [5] M. Mardani, G. Mateos, and G. Giannakis, “Dynamic anomalography: Tracking network anomalies via sparsity and low rank,” *J. Sel. Topics in Sig. Proc.*, Feb 2013.
- [6] V. Chandrasekaran, S. Sanghavi, P. A. Parrilo, and A. S. Willsky, “Rank-sparsity incoherence for matrix decomposition,” *SIAM Journal on Optimization*, vol. 21, 2011.
- [7] C. Qiu and N. Vaswani, “Real-time robust principal components’ pursuit,” in *Allerton Conference on Communication, Control, and Computing*, 2010.
- [8] C. Qiu, N. Vaswani, B. Lois, and L. Hogben, “Recursive robust pca or recursive sparse recovery in large but structured noise,” *under revision for IEEE Tran. Info. Th.*, also at *arXiv: 1211.3754[cs.IT]*, shorter versions in *ICASSP 2013 and ISIT 2013*.
- [9] C. Qiu and N. Vaswani, “Recursive sparse recovery in large but correlated noise,” in *48th Allerton Conference on Communication Control and Computing*, 2011.
- [10] J. He, L. Balzano, and A. Szlam, “Incremental gradient on the grassmannian for online foreground and background separation in subsampled video,” in *IEEE Conf. on Comp. Vis. Pat. Rec. (CVPR)*, 2012.
- [11] C. Hage and M. Kleinstueber, “Robust pca and subspace tracking from incomplete observations using l0-surrogates,” *arXiv:1210.0805 [stat.ML]*, 2013.
- [12] A. E. Abdel-Hakim and M. El-Saban, “Frpca: Fast robust principal component analysis for online observations,” in *Pattern Recognition (ICPR), 2012 21st International Conference on*. IEEE, 2012, pp. 413–416.
- [13] J. Feng, H. Xu, and S. Yan, “Online robust pca via stochastic optimization,” in *Adv. Neural Info. Proc. Sys. (NIPS)*, 2013.
- [14] J. Feng, H. Xu, S. Mannor, and S. Yan, “Online pca for contaminated data,” in *Adv. Neural Info. Proc. Sys. (NIPS)*, 2013.
- [15] E. Candès and T. Tao, “Decoding by linear programming,” *IEEE Trans. Info. Th.*, vol. 51(12), pp. 4203 – 4215, Dec. 2005.
- [16] G. Grimmett and D. Stirzaker, *Probability and Random Processes*. Oxford University Press, 2001.
- [17] E. Candès, “The restricted isometry property and its implications for compressed sensing,” *Compte Rendus de l’Academie des Sciences, Paris, Serie I*, pp. 589–592, 2008.
- [18] C. Davis and W. M. Kahan, “The rotation of eigenvectors by a perturbation. iii,” *SIAM Journal on Numerical Analysis*, vol. 7, pp. 1–46, Mar. 1970.
- [19] R. Horn and C. Johnson, *Matrix Analysis*. Cambridge University Press, 1985.
- [20] P. Feng and Y. Bresler, “Spectrum-blind minimum-rate sampling and reconstruction of multiband signals,” in *IEEE Intl. Conf. Acoustics, Speech, Sig. Proc. (ICASSP)*, vol. 3, 1996, pp. 1688–1691.
- [21] I. F. Gorodnitsky and B. D. Rao, “Sparse signal reconstruction from limited data using focuss: A re-weighted norm minimization algorithm,” *IEEE Trans. Sig. Proc.*, pp. 600 – 616, March 1997.
- [22] S. S. Chen, D. L. Donoho, and M. A. Saunders, “Atomic decomposition by basis pursuit,” *SIAM Journal on Scientific Computing*, vol. 20, pp. 33–61, 1998.
- [23] E. Candès, J. Romberg, and T. Tao, “Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information,” *IEEE Trans. Info. Th.*, vol. 52(2), pp. 489–509, February 2006.
- [24] D. Donoho, “Compressed sensing,” *IEEE Trans. on Information Theory*, vol. 52(4), pp. 1289–1306, April 2006.
- [25] E. Candès and T. Tao, “The dantzig selector: statistical estimation when p is much larger than n,” *Annals of Statistics*, 2006.
- [26] J. A. Tropp, “User-friendly tail bounds for sums of random matrices,” *Foundations of Computational Mathematics*, vol. 12, no. 4, pp. 389–434, 2012.